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AN ELEMENTARY TEXT-BOOK

OF

THEORETICAL MECHANICS

BY

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THEORETICAL MECHANICS.

W. P. I

PREFACE

This book is intended for upper classes in secondary schools and for the two lower classes in college. It is almost alone in its field, so far as American publications are concerned. The teaching of Mechanics as a subject *per se* has been confined in the main to collegiate courses, and the few American text-books on the subject have been written for students familiar with the calculus. At the same time, the few pages usually allotted to Mechanics in the elementary text-books of Physics are grossly inadequate for any substantial purpose. There is need of giving greater prominence to this subject in secondary schools, especially in institutions whose graduates look forward to industrial careers, either directly after graduation or through a subsequent course of engineering in college. The engineering student who has not studied Mechanics in his preparatory course should make up the deficiency during his freshman or sophomore year in college; it is a disadvantage to undertake the study of Analytic Mechanics without having previously covered the subject in a more elementary form — without the calculus. The especial needs of such students have been considered in the treatment of certain topics in this book.

As this is a text-book and not a treatise or a history of Mechanics, it is written from the standpoint of the student, without attempting to force upon him any particular sequence of topics and ideas that an analysis of the subject from other points of view might seem at times to require. Beyond a constant effort to abide by the fundamental precepts of teaching, no one method of presentation has been used to the exclusion of others. Only a few experiments are required or suggested. Any good teacher, however, could easily arrange a parallel course of laboratory exercises. In his own classes the author has found that the average student has acquired from his everyday observations and experiences an acquaintance with facts and phenomena quite sufficient to enable him to master the subject without a formal laboratory course. It is probably the same in the higher classes of all schools that include shopwork and general physics in their curricula.

Even in the purely academic high schools the course of general physics includes, as a rule, laboratory exercises on the simple machines, friction, acceleration, etc.

Each topic is followed by a few examples, but it has been deemed better not to insert a too liberal supply. Such examples, like laboratory exercises, are within the reach of, and can easily be devised by, any good teacher to meet his special needs.

A number of pages are devoted to the static treatment of Force, with emphasis on the idea of Action and Reaction. It is the author's opinion that it would be well to restrict the meaning of "action and reaction" to the conception of the dual nature of a force, — at any rate, to recognize more specifically the difference between a counteraction and a reciprocal action. The expression "force of inertia" is also objectionable. Inertia and friction are both passive resistances, — mere recipients of action, — and to regard them as such serves every purpose. Insistence on finer shades of meaning in this connection seems fairly demanded, on pedagogical grounds at least.

The four-place tables of the natural trigonometric functions, at the end of the book, are taken from Phillips and Strong's "Elements of Trigonometry," with the kind permission of the publishers of that book.

Professor William J. Raymond of the University of California read the manuscript of the last four chapters and offered numerous helpful suggestions, for which the author takes this occasion to express his hearty appreciation. My thanks are also extended to Mr. Fred H. Tibbetts, instructor of mechanics in the California School of Mechanical Arts, for reading the proof sheets.

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SAN FRANCISCO.

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THEORETICAL MECHANICS

INTRODUCTION

THE province of Theoretical Mechanics is indicated in the following outline of its subdivisions:

I. The branch called **Kinematics** is a study of the motions of bodies without reference to the cause of their motion; it inquires *how*, or in what manner, a body moves, as distinguished from *why* it moves.

II. When a force acts upon a body, it tends to set it in motion, or to alter its motion if it is already moving. But if there are different influences acting at the same time and these exactly balance each other, then the body is said to be in equilibrium, and the branch of mechanics treating of such cases is called **Statics**.

Pressures in fluids give rise to special considerations, which are grouped under a subdivision of statics called **Hydrostatics**.

In the study of statics, geometric and graphical methods are applied so widely, as in computations pertaining to roof trusses, bridges, etc., that it has been found convenient to give this mode of treating the subject a special name, **Graphical Statics**.

III. **Kinetics** is that branch of mechanics which discusses the forces acting upon bodies when motion is the result. It considers *why* the body moves, and the relation between the motion produced and the force or forces involved.

It will be observed that both statics and kinetics deal with forces. For that reason these two branches are sometimes classed together under the name of **Dynamics**, mean-

ing the science of forces, in contrast with kinematics, the study of pure motion.

The science of mechanics has its basis in certain fundamental facts and phenomena of nature, such as gravitation. The natural laws which these things seem to obey were not discovered in an instant by the stroke of a single mind, but have been worked out century after century through patient experiment and research. The ancients had glimpses, more or less hazy, of some of the now recognized laws of statics, and contributions of permanent value were made by Archimedes before 200 B.C.; but mechanics as a science rests historically and philosophically on the principles of kinetics enunciated by Galileo, Newton, and other profound physicists and mathematicians of the sixteenth and seventeenth centuries. Their successors, with the aid of improved mathematical methods and processes, have extended and simplified these laws and principles, and brought them together in the form of a comprehensive subject of study.

In an elementary treatment of mechanics the mathematical operations are mainly arithmetical, but are greatly facilitated by the use of algebra, geometry, and trigonometry. Most of the units employed in these operations, as in the measurements of length, area, volume, and capacity, and also in angular measurements, are familiar, and, as a rule, are sufficiently clear to the average person. But in some instances the everyday meaning attached to a word is not accurate enough for scientific purposes. A precise definition of **Mass** is rarely required or attempted outside of mechanics and other branches of physics, and in varying degrees the same may be said of the related ideas of weight, density, and specific gravity. Likewise, the idea of **Force**, in its scientific conception, is seldom conveyed by this word as used by most persons.

The unit of **Time**, a second, is well understood as meaning a certain fraction of a day, but it should be remembered that

a "day" is not always of fixed length. If the earth had but one motion, a rotation upon its axis, or, if its orbit were circular, the successive transits of the sun across any meridian (noonday), would always occur at equal intervals. But since its orbit is elliptical rather than circular, the interval between transits is not constant. For scientific accuracy this difference must sometimes be allowed for; but for most purposes it is sufficient to take as the length of a "mean solar day" the average of these successive intervals of transit of the sun across any meridian. The ordinary second is $\frac{1}{60 \times 60 \times 24}$ of this mean solar day.

Just as the letters of the English alphabet are used in algebraic and geometric notations, so at times the Greek letters serve as helpful symbols, particularly for marking angles. The most familiar is the letter π, which is reserved for a special use: to represent the numerical value of the ratio of the circumference of any circle to its diameter. A few others are used more or less conventionally with special meanings, but as a rule no such discrimination is necessary. Following are the small (or, as a printer would say, the "lower case") letters of the Greek alphabet, with their English equivalents and pronunciations:

Greek	English Equivalent	Name	Greek	English Equivalent	Name
α	a	Alpha	ν	n	Nu
β	b	Beta	ξ	x	Xi
γ	g	Gamma	o	o (short)	Omicron
δ	d	Delta	π	p	Pi
ϵ	e (short)	Epsilon	ρ	r	Rho
ζ	z	Zeta	σ	s	Sigma
η	e (long)	Eta	τ	t	Tau
θ	th	Theta	υ	u	Upsilon
ι	i	Iota	ϕ	ph	Phi
κ	k	Kappa	χ	ch	Chi
λ	l	Lambda	ψ	ps	Psi
μ	m	Mu	ω	o (long)	Omega

As it will be necessary to convert values back and forth between the English and metric systems of weights and measures, keep in mind the following approximate relations:

1 meter	= 39.37 inches.
1 inch	= 2.54 centimeters (cm.).
1 cubic centimeter (c.c.)	= 0.061 cubic inches.
1 cubic inch	= 16.39 cubic centimeters.
1 kilogram (Kg.)	= 2.2 pounds.
1 liter	= 1.06 quarts.

SECTION I

KINEMATICS

CHAPTER I

MOTION. VELOCITY

Velocity. — When a body moves from a given point, a full description of its motion involves:

1. Its direction of motion.
2. Its rate of motion.

By the rate of motion we mean the distance traveled in a unit of time. This is usually called the **velocity** of the body, although the rate of motion, when considered independently of the direction, is sometimes called the **speed** of the body. Technically the word "velocity" is the broader term, because it may comprehend the direction of motion as well as its rate.

EXAMPLES

1. (*a*) *A velocity of 10 meters per second is equivalent to how many feet per second?*

(*b*) *Express the same velocity in feet per minute.*

2. (*a*) *Express in feet per second a velocity of 100 meters per minute.*

(*b*) *Express the same velocity in feet per hour.*

3. (*a*) *A velocity of 1000 meters per minute is equivalent to how many miles per hour?*

(*b*) *Express the same velocity in feet per second.*

4. *Which velocity is the greater and by how much, — 40 miles per hour, or 12 meters per second?*

Uniform Motion; Average Velocity. — The distance traveled in any given time by a body moving with uniform velocity is expressed by the formula

$$d = vt. \tag{1}$$

If t is expressed in *seconds* and d in *feet*, then v must be in *feet per second.* In using any formula it is always necessary to know in what unit or units each of its component quantities is expressed, and all should be in the same system.

By transformation this formula becomes

$$v = \frac{d}{t}, \tag{2}$$

which indicates that the velocity of any moving body may be found by dividing the distance traveled by the units of time consumed. (Compare this statement with the definition of velocity already given.) This is true even if the speed of the body is not uniform, the quotient d/t representing in this case the average velocity during the time under consideration.

EXAMPLES

1. *A train travels 10 miles at a rate of 20 miles per hour; then 4 miles at an average rate of 30 miles per hour; then 6 miles at a uniform rate of 40 miles per hour; then it comes to rest after traveling a distance of 1 mile (slowing down), running meanwhile at an average rate of 20 miles per hour; it stands for 7 minutes; then it starts and runs for 20 minutes at an average rate of 21 miles per hour. What has been its average velocity for the entire time?*

2. *The velocity of sound in air is about 1090 feet per second, and the velocity of light is 186,000 miles per second. If I see a flash of lightning 5 seconds before I hear the report of the thunder, how far away is the disturbance?*

Relativity of Motion. — A body at rest on the earth's surface moves with the earth in diurnal rotation and progresses with

the earth in its path of revolution around the sun. There are other minor motions of the earth, and furthermore it is sweeping through space along with the sun and other members of the solar system. But ordinarily it is not necessary to take into consideration any of these when we speak of the motion of a body. Unless otherwise specified we assume that a body is at rest if it is not changing its position relatively to other bodies on the earth's surface, and in the same manner we usually describe the motion of a body by noting its change of position relatively to other fixed terrestrial objects. When we say that a body is moving toward the northeast or in any other direction of the compass, we assume that it remains in a horizontal plane and we are really describing its motion by reference to two axes at right angles to each other, one a north-and-south line or meridian and the other an east-and-west line or parallel of latitude. By the aid of a third or vertical axis at right angles to the other two we can further describe the motion of any moving body that does not remain in a horizontal plane. When we speak of the motion of a body relatively to the earth, the description is not complete unless we make clear in all respects the direction of motion.

There are times, however, when the motion of a body must be considered relative to objects other than the earth, but in every case, so far as we know, motion is relative. In astronomical work all the complexities of the earth's motion are taken into consideration one way or another, by measuring all motions relatively to the fixed stars. A person riding in a closed vehicle may either knowingly or unconsciously assume planes or points of reference within the vehicle. Or, we may inquire concerning the motion of a body relatively to another entirely independent body that is also moving; for instance:

Two trains pass each other at a station, called A. One train, called E, is moving eastward at the rate of 30 miles per hour; the other, W, is moving westward 25 miles per hour.

Find: (*a*) *The velocity of E relative to A.*
(*b*) *The velocity of W relative to A.*
(*c*) *The velocity of E relative to W.*
(*d*) *The velocity of W relative to E.*

In this case, if we consider only the *rate* of motion, then, since the train E would be located 30 miles from the station at the end of an hour, it would be sufficient to say that its velocity relatively to the station is 30 miles per hour. But if we also consider *direction* of motion, then we must add that the motion of the train E relatively to the station is in an eastward direction. The velocity of the station relatively to the train E is 30 miles per hour westward.

Likewise the motion of W relatively to A is 25 miles per hour westward.

At the end of the hour the train E is 55 miles east of the train W. Without giving to the train W any credit for its part of the transaction, we simply recognize the fact that in the hour of time E reaches a position of 55 miles east of W, and we express this by saying that the velocity of E relatively to W has been 55 miles per hour eastward.

Conversely, we must say of W, that its velocity relatively to E has been 55 miles per hour westward.

EXAMPLE

A train 400 feet long travels between two mileposts at a uniform rate of 30 miles per hour.

FIG. 1

At position $\boxed{1}$ *a person starts from the last car to walk through the train, and reaches the front at the instant the train reaches* $\boxed{2}$.

(*a*) *With what average velocity in miles per hour did he walk?*

(*b*) *What was his average velocity relatively to the ground?*

(*c*) *What would have been the answers to* (*a*) *and* (*b*) *if he had walked to the back from the front of the train?*

Graphical Representation of Motion. — The motion of a body at any instant — both the magnitude and the direction of its velocity — can be represented by a straight line. For this purpose we may adopt any convenient scale of magnitude and any arbitrary notation to indicate direction. For instance, a velocity of 20 miles an hour in an eastward direction could be represented by a line 2 inches long (1 inch = 10 miles per hour), drawn horizontally from left to right from the starting point, the direction being indicated by an arrow, as OA in the following figure:

$$O \xrightarrow{\qquad\qquad\qquad\qquad} A$$

FIG. 2

If in connection with this motion we wish to refer to a second motion, — say a velocity of 15 miles an hour northward from the same point, — we would have to adhere to the same scale (1 inch = 10 miles per hour), and conventionally we should draw this line vertically upward.

EXERCISES

1. *Using the same notation, represent each of the following motions, all starting from the same point:*

22 miles per hour southward.
5 miles per hour westward.
17 miles per hour N. E.
14.5 miles per hour S. E.
3 miles per hour 65° S. of E.
20.5 miles per hour 77° N. of E.
11.7 miles per hour 14° W. of N.
11.7 miles per hour 33° W. of S.

It should be carefully noted that the scale of magnitude in this figure is not a scale of distance, and the lines themselves do *not* represent *distances* merely, *but velocities*. A similar method could be, and frequently is, used for distances, and hence it becomes necessary to keep clearly in mind the real signification of each

diagram, and each line in it. Take, for example, the confusion that might arise in interpreting a diagram constructed as in the next exercise.

2. *On a scale of 1 inch = 10 meters, construct a diagram to represent the* PATH *of a body moving as follows: From the starting point the body moves westward 10 seconds, at the rate of 3 meters per second; thence northward 2.4 seconds at the rate of 5 meters per second; thence northeast 7 seconds, at the rate of 1.7 meters per second; thence 30° east of south for 6 seconds at the rate of 2 meters per second.*

Having constructed this diagram, we can use it for any measurement of *distances* concerned with the path of this body. For instance, at the end of the motion how far was this body from the starting point? But if we wish to consider incidental questions of velocity, it becomes necessary to introduce additional computations involving the consideration of *time*. For example, if a second body had moved in a straight line from starting point to finish, instead of by the roundabout path, at what rate must it have traveled in order to arrive at the terminal point simultaneously with the first body?

It is true that a line which represents a velocity may be regarded at the same time as representing distance, provided we remember that it is *distance per unit of time*.

CHAPTER II

COMPOSITION OF VELOCITIES

SOMETIMES, and generally, the actual motion of a body is the result of several different influences acting simultaneously, as in rowing a boat in a tidal current. Heretofore, in speaking of the velocity of a body, we have thought of it only as a simple motion in one direction, taking it as we find it, no matter how many different influences may have contributed to give the body this velocity and path. When two or more influences act *simultaneously* to produce motion in a body, each influence has its full effect in its own direction, as if the other influences did not exist. The motion that each of these influences would produce is called a **component**, and the actual motion, as we have been considering it, is called the **resultant** of all these components. There are times when the component motions are known and the resultant cannot be found except by computing it from the components; at other times the actual motion is given, and it is desired to find its component parts. We have only to take the literal meaning of "component" to see that the resultant is obtained by "putting together," or combining, these more elementary motions; but the mathematical process of doing this is not always simple, although sometimes it is a mere addition or subtraction.

For the convenience of the beginner it is well to assume three cases, exemplified as follows:

(i) If a person walks directly forward through a moving car, his resultant velocity is his rate of walking *plus* the

velocity of the train. This would be the composition of motions parallel to each other.

(ii) If he walks straight across the car, his resultant velocity is the hypothenuse of a right triangle of which one side represents the rate at which he walks and the other side represents *on the same scale* the velocity of the train. This is illustrated in Fig. 3. The man starts from A to move across the car towards the window W. Suppose that in a unit of time he covers that part of the distance represented by AB; then the arrow AB represents his velocity of walking. Now, since the car is in motion, suppose that it progressed

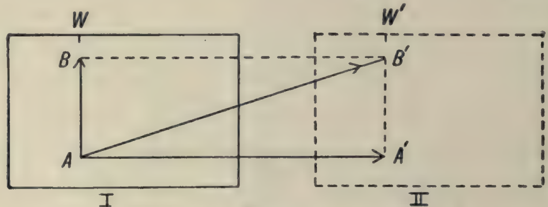


FIG. 3

from position I to position II while the man walked from A to B, that is, in a unit of time. Then the line BB' will represent the velocity of the car, or any point in it, as B. Hence, at the end of the unit of time the man finds himself, not at B (as he would if the car were at rest), but at B', and the actual path he has traveled in space — or, better, relatively to the ground — is AB'. This hypothenuse, therefore, represents his resultant velocity, on the same scale as AB and AA'.

Properly speaking, the two components of AB' are the two motions of the man himself, — one AB, relatively to the car (which also would have been his only motion relatively to the ground if the train had not been moving when he walked across); and the other AA', which would have been his only motion relatively to the ground had he stayed in his seat while the train moved. These two motions occur-

ring conjointly, their resultant has a direction between the two. In finding the magnitude and direction of AB', it makes no difference whether we use the triangle ABB', or its equal $AA'B'$; each is half the rectangle $ABB'A'$.

(iii) If he walks obliquely across the car towards a point X, as in Fig. 4, his resultant motion relatively to the ground will be the side AB' of the obtuse triangle ABB'. This case differs from the last only in the obliqueness of the two components AB' and AA', but this difference is a very important one from a mathematical point of view. It is only when the angle BAA' is 90°, or 45°, or 60°, or 120°,

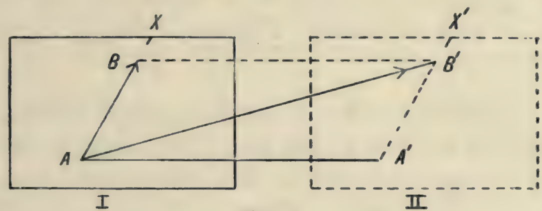


FIG. 4

that the triangle ABB', or $AA'B$, can be readily solved without using trigonometry.

Composition of two motions parallel to each other. — In the example on page 14, as the man walks through the train, he is subject to the additional motion of the train itself. In part (*b*) of the problem he covers in two minutes a distance of 400 feet (by walking) + 5280 feet (by the progress of the train). These two component motions take place simultaneously, and since they are in the same direction the resultant velocity of the man relatively to the ground is the sum of the two components; $v=d/t=\frac{5280+400}{2}$ feet per minute.

Observe that the two motions occur in the same interval of time; if they had been *consecutive* instead of *concurrent*, the conditions would have been very different and would not have come within the scope of "composition of velocities." That is, if the man had remained in his seat at the rear of the train until the

train came to rest at mile post $\boxed{2}$, he would have covered a distance of only 5280 feet in the two minutes; if, thereafter, he had walked to the front of the train while it was still in waiting at mile-post $\boxed{2}$, he would have required two minutes additional time in which to walk the length of the train, 400 feet. His velocity during the first two minutes would have been $\frac{5280}{2}$ feet per minute, and during the second two minutes, $\frac{400}{2}$ feet per minute. If there were any need to know his *average velocity* during the total of these two intervals of time, it would be $\frac{5280+400}{4}$ feet per minute, but his motion during the second two minutes has no effect to change the rate at which he covered ground during the first two minutes, and *vice versa*. In other words the "composition of velocities" is concerned only with the resultant of two or more components, acting conjointly or simultaneously.

The resultant of two or more **parallel motions** in the same direction is equal to the sum of the components; if the two motions are in opposite directions, their resultant is equal to their difference. Using the signs + and − to indicate opposite directions, we can say, in general terms, that the resultant of two parallel motions is equal to their algebraic sum.

A person rowing directly with or against wind and tide is subject to three influences, each of which, if undisturbed, would cause him to move with a certain velocity. His resultant or actual motion could be found by summing up the three motions that would have been occasioned by these influences acting singly.

EXAMPLES

1. *In the example on page 14, if the man had walked to the rear of the train from the front, as in part* (*c*), *what would have been his resultant velocity?*

2. *In Example 2, page 16, were the motions designated as simultaneous, or as consecutive?*

Components at Right Angles. — The statements in (ii), page 18, will aid in the solution of the following:

EXAMPLES

1. *If a man undertakes to row straight across a channel in which there is a current, his course will be oblique.*

Let AB = his velocity of rowing,
and AC = his velocity due to the current.

Then the resultant AD, the diagonal of the parallelogram, will be the direction of his course.

(a) If $AB=4$ miles per hour, and $AC=3$ miles per hour, what is his resultant velocity?

(b) If the DISTANCE *straight across the channel (AE) is eight miles, what is the distance AF.*

How long does it take him to reach F?

How long would it have taken him to reach E, if there had been no current?

(c) Find the value of the angle DAB by trigonometry.

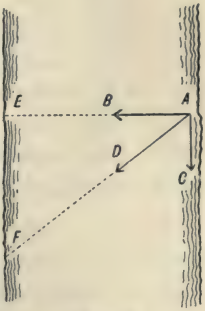


FIG. 5

2. *If he wants to go straight across, he must head the boat upstream, as $A'B'$,* Fig. 6.

Let $A'C'=3$ miles per hour, as before, and let the DISTANCE *$A'E'=8$ miles.*

In what direction and with what velocity must he row in order to reach E', directly opposite A', in two hours?

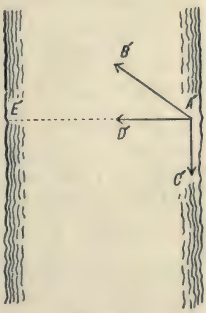


FIG. 6

3. *A boat B, 300 yards from shore and 100 yards upstream from A, Fig. 7, is carried downstream 3 miles per hour.*

(a) If it is kept headed perpendicular to the shore, with what velocity must it be rowed in order to land at A?

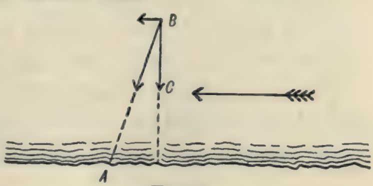


FIG. 7

(b) Find resultant velocity, and magnitude of the angle ABC.

4. *A steamship S, 6 miles from the shore, AB, is supposed by those on board to be sailing from south to north 10 miles per hour, but really is also drifting toward shore at the rate of 2 miles per hour. The shore extends north and south.*

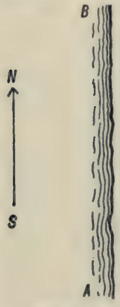


FIG. 8

(a) How long before the steamship will reach the shore? Designate the point at which it will strike. With what velocity will it strike?

(b) If it is desired that the steamship shall progress exactly northward at the rate of 10 miles per hour, in what direction and with what velocity must it steam in order that the action of its engines combined with the influence of the current may give it the desired resultant motion?

Note how these examples illustrate the statement, previously made, that when two or more influences act simultaneously to produce motion in a body, each influence has its full effect in its own direction, as if the other influences did not exist.* In Example 4 (*a*) it will be noticed that the ship drifts toward the shore at a certain fixed rate, no matter how fast it is sailing in a direction parallel to the shore; and conversely, its progress from south to north is not influenced by the drifting shoreward. Likewise, in Example 1 the current does not retard the man's progress across the channel, but simply carries him downstream.

Two Components at Any Angle. — This is the general case and requires the use of trigonometry for solution, although results sufficiently reliable for most purposes can be obtained by graphical methods from carefully constructed diagrams.

Suppose, in a given case, that a body is subject to two simultaneous motions, one 120 feet per second and the other 70 feet per second, the directions of the two differing by 60°. For graphical representation assume a scale of 1 inch = 100 feet per second. Let O be the starting point of the body. Draw OA (Fig. 9) equal to 1.2 inches, to represent one of the velocities, and OB, equal to

* This principle was first announced by Galileo, who recognized its importance when he was studying the question of projectiles, dealt with on pp. 59 to 65 of this book.

0.7 inch, to represent the other, the angle or difference of direction between them being made equal to 60°. From point A as a center describe an arc whose radius is equal to OB, and from B as a center, with radius equal to OA, describe a second arc intersecting the first at C. The figure $OACB$ is thus a parallelogram. Its diagonal OC represents the resultant of OA and OB in both magnitude and direction, on a scale of 1 inch = 100 feet per second.

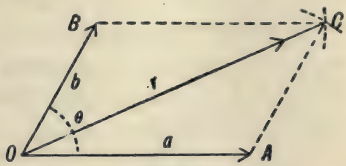


FIG. 9

This device or construction was made use of by Stevinus (1600), and afterward (1687) the general principle was fully explained by Newton and Varignon.

Calling these components a and b respectively and the resultant r, it can be shown by trigonometry that

$$r = \sqrt{a^2 + b^2 + 2\,ab\cos\theta}, \qquad (3)^*$$

where θ is the angle between a and b.

It can also be shown that this formula is the same whether θ be acute or obtuse.

EXAMPLES

1. *A boat sailing westward 13 miles per hour is carried by the tide in a direction 23° W. of S. at the rate of 3.5 miles per hour. Find its resultant motion (velocity and direction).*

2. *A bird flying in a N. E. direction at the rate of 28 miles per hour encounters a wind blowing from a direction 12° W. of N. which carries him out of his course (that is, in the direction of the wind) at the rate of 19 miles per hour. Find his resultant velocity and direction.*

3. *In the formula* $r = \sqrt{a^2 + b^2 + 2\,ab\cos\theta}$, *if* $\theta = 180°$, *what is the value of* r*? What is the value of* r *when* $\theta = 0°$*? When* $\theta = 90°$*?*

* It should be carefully noted why this formula differs from the formula for the trigonometric solution of a triangle, given two sides and their included angle.

Resolving a Velocity into Components.

EXAMPLE

If a body is moving in a direction exactly N. W., 4.5 miles per hour, how fast is it progressing northward, and how fast westward? Solve this by constructing a diagram to scale, and also by trigonometric computation.

In this instance we have assumed the given velocity to be a resultant, and have resolved it into two components, one north and the other west. We might have resolved it into other components,—two components in directions other than north and west, or even into several components.

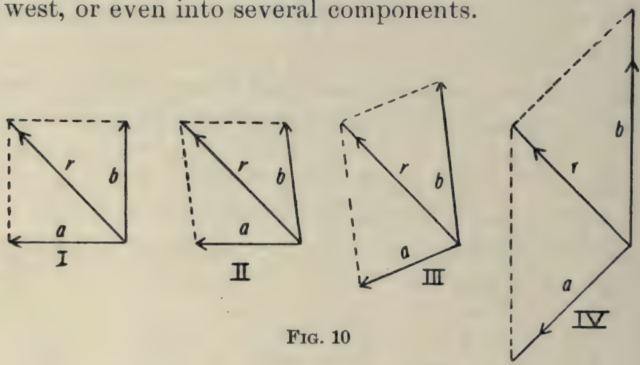


FIG. 10

Starting with the given velocity of 4.5 miles per hour in a N.W. direction, we can draw an infinite number of parallelograms of which this line will be a diagonal. In Fig. 10, the two sides *a* and *b* of any one of the parallelograms may be taken as the two components, of which the actual motion of 4.5 miles an hour in a N.W. direction is the resultant. Sometimes, as in the case of a ship sailing in an invisible current, it is utterly impossible to discover the true components which have operated to give the resultant motion, but as a rule sufficient conditions are known from which to determine what the elementary motions would have been if each of the component influences had acted singly.

To resolve a resultant into two components requires the solution of a triangle (as in the converse proposition of finding the resultant from the components), and hence at least *three* parts of the triangle must be known. If we start with the resultant given, the conditions of the problem must state, or imply (1) the lengths of the other two sides; or (2) one of these sides and an angle; or (3) two of the angles.

The resolution of a velocity into components was illustrated in Examples 2 and 3, and the last part of Example 4, pages 21 and 22.

This triangle to be solved always contains *one of the components* and *a side parallel and equal to the other component*, but it never contains both components. This fact frequently occasions confusion in the graphical representations that accompany this branch of kinematics, and leads beginners into many difficulties in gaining a clear conception of the relation between a resultant and its components. Remember that the resultant is always the diagonal of a parallelogram of which the components are two sides; and that the two components and the resultant all start from the same point.

EXERCISES

Solve by trigonometry. Also, as a check upon errors, construct diagram on convenient scale in each case. Those who have not studied trigonometry can get sufficiently reliable results from diagrams.

1. *A ship is sailing at the rate of 3 miles per hour and a sailor climbs the mast 100 feet high in 1 minute. What is the magnitude and what the direction of his resultant velocity?*

2. *A balloon leaving the ground ascends with a vertical velocity of 40 feet per second and is carried with the wind. If it rises at an angle of 83° with the horizon, at what rate is it moved by the wind?*

3. *One of the rectangular components of a velocity of 60 miles per hour is a velocity of 30 miles per hour; find the other component.*

4. *The components in two directions of a velocity of 30 miles per hour are velocities of 15 and 25 miles per hour; determine their directions.*

5. *A steamship is headed in a direction 38° S. of E. in a wind blowing from a direction 27° 30′ E. of N. If the ship's progress proves to be in a direction 43° S. of E. with a velocity of 17 miles per hour, at what rate was it steaming, and at what rate was it carried with the wind?*

6. *Find the horizontal and vertical components of the following velocities:*

(1) *1000 feet per second in a direction inclined 30° to the horizon.*

(2) *The same velocity in a direction inclined 50° to the horizon.*

(3) *25 miles per hour at 60° to the vertical.*

7. *Find the magnitude and direction of the resultant of the following motions (four components), to which a body is subjected simultaneously:*

10 feet per second, E.; 7 feet per second, N.; 13 feet per second, W.; and 16 feet per second, S.

8. *Solve by geometry:*

(*a*) *Given two equal components at an angle of* $\beta = 60°$. *Find the resultant.* (HINT: The diagonals of a rhombus bisect each other at right angles.)

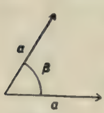


FIG. 11

(*b*) *Find the resultant of two unequal components,* a *and* b, *at an angle of 60°.*

(*c*) *Find the resultant of two equal components at an angle of 120°.*

(*d*) *Find the resultant of two unequal components at an angle of 120°.*

9. *In Formula 3, page 23, substitute for* θ *the values 60° and 120°, and compare with the geometric results of Example 8, for both equal and unequal components.*

10. **The Ship Problem.** — *A ship is pointed W. to E., with her mainsail set at an angle of 20° (δ in diagram), Fig. 12 a. The wind*

blows at an angle of 75° with the sail (indicated by β in diagram), with a velocity of 15 miles per hour.

(*a*) *What is the component of wind velocity in the direction of the length of the ship?*

Discussion. — This question is not intended to imply that the motion of a ship is caused to any great extent by the component of the wind velocity parallel to the direction in which the ship is headed. The main motive power comes from the action of the wind on the sail; if we consider only the wind components parallel and perpendicular to the hull, we are dealing merely with minor influences, — one acting on the stern of the hull, pushing the ship forward, and the other acting at right angles to the hull and pushing the ship out of her course. Although these influ-

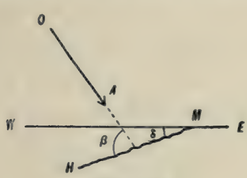


Fig. 12 *a*

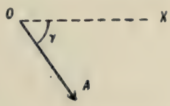


Fig. 12 *b*

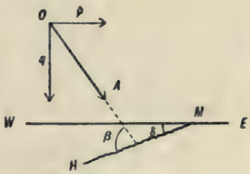


Fig. 12 *c*

ences have little to do with propelling the ship, still, it is well to study them and put them aside before considering the action of the wind on the sails.

Notice, also, that the question asks only for the wind component in the direction of the ship. This cannot be found from the conditions of the problem unless we assume something concerning the other component. The solution of a triangle requires that at least three parts be given. In this case if OA, Fig. 12 *b*, represents the wind velocity in magnitude and direction, and a horizontal line, OX, is parallel to the direction of the ship, we have as given conditions only the line OA and the angle γ. The magnitude of the horizontal component may be almost anything, — depending upon what conditions are attached to the other component. (See pp. 24 and 25.)

When not otherwise specified, it is customary to take the two components at right angles to each other. In this case, when

asked to find the component parallel to the direction of the ship, we have assumed that the other component is at right angles to the ship (Fig. 12 *c*).

(*b*) *What are the wind components parallel and perpendicular to the sail (Fig. 13 a)?*

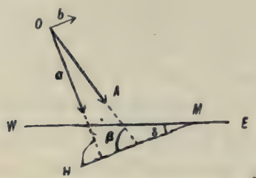


FIG. 13 *a*

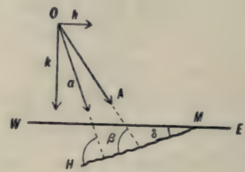


FIG. 13 *b*

(*c*) *The component* a, *pushing squarely against the sail, is not entirely useful in moving the ship forward ; on the contrary, the ship tends to move bodily in the direction of this component. How much is component* a *acting in the direction of the ship, and how much is it acting perpendicular to that direction ?* (*See Fig. 13 b.*)

11. The Kite Problem. — *A kite flying in a wind blowing 12 miles per hour is inclined at an angle of 58° with the horizon, as angle β in Fig. 14 a.*

(*a*) *Assuming that the wind is blowing horizontally, what are its components* — a *perpendicular, and* b *parallel, to the kite ?*

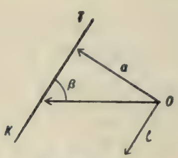


FIG. 14 *a*

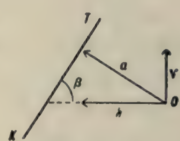


FIG. 14 *b*

(*b*) *The component perpendicular to the kite represents the total pull exerted on the string (assuming that the string is perpendicular to the kite). A part of this pull is upward and a part horizontal. Find each part,* h *and* v, *Fig. 14 b.*

The part *v*, determined in this manner, is the lifting component of the wind on the kite.

12. *If a body travels* d *miles in* t *hours, what is its velocity in feet per second?*

13. *If a body has a velocity of* v *feet per second, in how many hours will it travel a distance of* m *miles?*

14. *Find the resultant of two perpendicular components, one* p *meters per minute, and the other* q *feet per second.*

"Triangle of Velocities" and "Polygon of Velocities." — These two ideas find frequent expression in mechanics. The former has been referred to incidentally in a previous paragraph, and is now reverted to for the purpose of directing attention to a consideration that will serve to facilitate the extension of the subject of composition of velocities to cases in which there are more than two components.

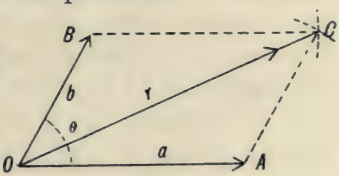


FIG. 15

As already stated, the resultant of two components a and b is one of the diagonals r of a parallelogram $OACB$ (Fig. 15). In problems concerned with a, b, and r, it is customary to solve one of the triangles OBC or OAC. If we work from the triangle OBC, we do not deal directly with component a, but with BC, parallel and equal to a; or, if we choose to work from triangle OAC, we consider AC in place of component b.

What is signified by this substitution of a line parallel and equal to one of the components for the component itself? It has already been emphasized (p. 19) that the very idea of the composition of velocities presupposes that the component actions take place simultaneously. In the figure of the parallelogram this conception of simultaneous action is easily comprehended. It is abandoned, however, for the time being as soon as we transfer our thoughts to one of the triangles alone; for instance, if we take the triangle OAC apart from the rest of the parallelogram, the resultant is found

mathematically from OA and AC, Fig. 16, as if resulting from these two motions taken consecutively, whereas, in truth the component motions not only are *not consecutive*, but AC is not one of them at all. This misconception grows out of the tendency, already referred to, to assume that OA and AC represent *distances* or *displacements*, instead of *velocities*. The "triangle of velocities" should be used with a clear understanding of the meaning of each line.

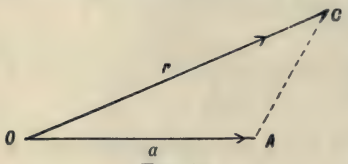


FIG. 16

As a mathematical expedient, the "triangle of velocities" is a simple case of a more general construction, the **polygon of velocities**, which suffices not merely for two, but for any number of components. The idea is simple. Let p and q (Fig. 17 a) be two components with resultant r. If we draw a line OA (Fig 17 b) to represent p; from A, a second line AC parallel and equal to q; and thence from C a line CO to the starting point, we shall have constructed a closed figure of

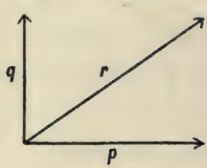


FIG. 17 a

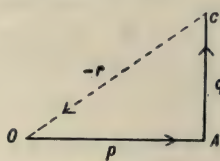


FIG. 17 b

which the closing line is *parallel and equal to* the resultant r, but drawn in the *opposite direction*. For the sake of convenience, we temporarily lay aside all thought of simultaneous action and consider the components successively, and we thereby reach a conclusion which we know bears a certain relation to the resultant, — a line equal to the resultant but drawn in an opposite direction.

A similar device, really the same process, can be used for any number of components. Let p, q, s, and t (Fig. 18 a) be

four components. By resorting to the parallelogram method we can get the resultant r' of p and q; then we can treat r' as a single velocity and combine it with s in the same way, getting a new resultant r''; this in turn can be combined with t to obtain the final resultant r'''.

By using the **polygon of velocities** we could have determined this resultant much more readily. Draw a line equal and parallel to p (Fig. 18 *b*); from the extremity of this draw a second line equal and parallel to q; thence a line equal and

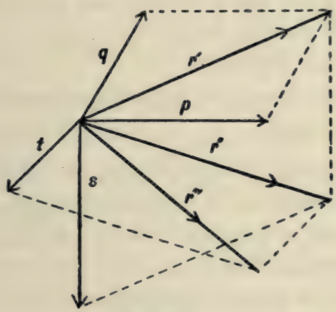


FIG. 18 *a*

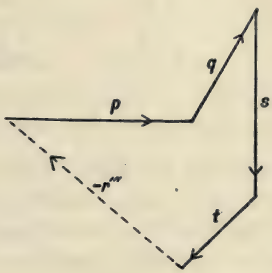


FIG. 18 *b*

parallel to s; and one equal and parallel to t. The line necessary to close this polygon will be *equal to the desired resultant but opposite in direction.*

While the "polygon of velocities" simplifies the graphical determination, it would still be necessary to break up the figure into triangles in order to accomplish a trigonometric solution.

EXAMPLES

1. *How would it have been if we had combined the components of Fig. 18 a in a different order? Try it, assuming any four velocities and finding their resultant by combining them graphically in at least two different orders by the parallelogram method.*

2. *If the lines of Fig. 18 b were constructed in a different order, would the result be the same? Try it.*

3. *Find the resultant motion of a body which has the following component motions:*

8 miles per hour from W. to E.;
3 miles per hour 34° W. of S.;
5.5 miles per hour 11° S. of W.

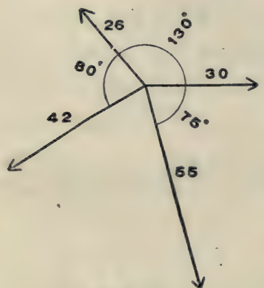


Fig. 19

Construct the "polygon of velocities" to accurate scale, and compare the graphical result with the trigonometric solution.

4. *A body has four motions measured in meters per second, as indicated by the numbers in Fig. 19.*

Find the resultant by construction and by computation.

Resultant for Angles Greater than 90°. — It is frequently asked, "How can the resultant be less than one of the components?" This is often the case when the angle between two components is greater than 90°. Let the two components be a and b (Fig. 20 a). For convenience, we may regard a as the resultant of the two

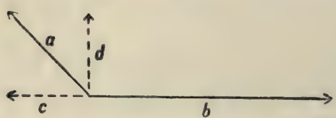


Fig. 20 a

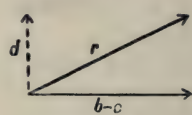


Fig. 20 b

components, c and d, at right angles to each other. From this it is evident that the component a is acting against b to some extent, for the reason that b, c, and d together are equivalent to b and a. Subtracting c from b, we have left a horizontal velocity represented by the difference between c and b, which, combined with d (Fig. 20 b), will give a resultant identical with the resultant of b and a. This is shown from the formula,

$$r^2 = a^2 + b^2 + 2\,ab \cos\theta.$$

If θ is greater than 90°, $\cos\theta$ is negative; and from 90° up to 180° the larger the angle the smaller the resultant. (See Example 3, p. 23.)

CHAPTER III

CIRCULAR MOTION

Motion of a Body in a Circle. — It has been stated (p. 11) that a full description of the motion of a body involves:

1. Its direction of motion.
2. Its rate of motion.

A body moving in a circle may have uniform speed, but its motion differs from the rectilinear motions heretofore considered by constantly changing in direction. It is only necessary to recall the familiar geometrical distinctions between straight lines, broken lines, and curved lines, in order to comprehend the difference between rectilinear and curvilinear motion, — the former constant in direction, and the latter constantly changing direction. Motion in a circle *at a uniform speed* is the simplest of curvilinear motions, because the *change of direction* takes place *at a uniform rate.*

The speed is found in the usual way, $v = d/t$. For instance, if a body is moving in the circumference of a circle with radius equal to r feet, and requires six seconds for completing one revolution, its velocity is $2\,\pi r/6$ feet per second. In lineal measure this will be $\frac{1}{6}$ of the length of the circumference, and in angular measure 60°, as shown in Fig. 21. If the radius had been twice as large, the body, moving at the same speed, would traverse an arc of only 30° in one second; and if the radius were half as large, the angular change would be 120°. The rate of angular change, therefore, depends upon the radius of the circle as well as upon the speed of the body; it is directly proportional to one and inversely proportional to the other.

Properly, this rate of angular change should be determined from tangents rather than from the radii, as illustrated in Fig. 22. When the body moving in the circle is at point A, its motion for the instant is in the direction of the tangent AT; at point B its motion is in the direction of the tangent BT'. If the arc AB is 60°, the angle TCT' is also 60°. If the body moves in the circumference from A to B in one second, it not only has a velocity of $2\pi r/6$ feet

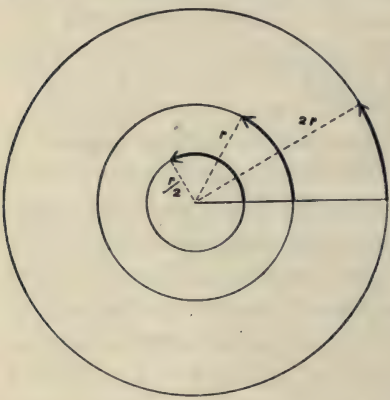


FIG. 21

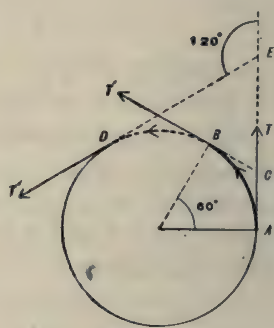


FIG. 22

per second, but the direction of its motion is also changing at the rate of 60° per second. At the end of two seconds the body will have covered a distance AD equal to $2\,AB$, and the total change of direction will be 120°.

The rate of angular change,

$$\frac{\textit{total change of direction}}{\textit{time}},$$

is called the **angular velocity**. In this sense the word "velocity" does not have its usual meaning of distance traveled per unit of time, but is given a much broader significance, equivalent to the general expression "time rate of change." This is a figurative use of the word that may be somewhat confusing to beginners. For example, "temperature velocity" would mean the rate at

which the temperature is changing. In the same way, "angular velocity" means the rate at which the direction (difference of direction, or angle) changes.

As already stated, angular velocity may be measured in degrees per unit of time, but in this connection it is customary to use, instead of degrees, a less common unit of angular measurement, called a **radian**. The reason for discarding the degree unit in this instance becomes obvious when we understand how the idea of the radian simplifies the calculations necessary in converting angular velocities into lineal measure. If we take any circle and on its circumference measure off a lineal distance equal to the radius of the circle, we get an arc which is a certain fraction of the total circumference. The angle at the center by which this arc is subtended is called a radian. The number of degrees in a radian is easily calculated. We know that the length of the circumference is $2\pi r$, or $2 \times 3.1416 \times$ length of radius. Hence there are 6.2832 radians in 360°, or one radian contains about 57° 18′, or 57.3°.

Suppose, then, that a body traveling in a circle of 6-inch radius covers 5 radians per second. With the angular velocity expressed in this way we can readily change it into degrees per second by multiplying by 57.3, or we can convert it into lineal velocity by multiplying by 6, the length of the radius. Hence, if we designate the angular velocity in radians per unit of time by the Greek letter ω (which is customary), and if r is the length of the radius of the circle, then the lineal velocity is $v = r\omega$. Or, conversely, if a body is moving in the circumference of a circle, radius r feet, with a velocity of v feet per second, then its angular velocity, ω, is equal to $\frac{v}{r}$ radians per second, or $\frac{v}{r} \times 57.3°$ per second.

In connection with machinery angular velocities are commonly expressed as "revolutions per minute."

EXAMPLES

1. *How many radians in a right angle?*

2. *If the earth's radius is 4000 miles, what is the length of the arc on the earth's surface which subtends an angle of one degree at the earth's center?*

3. *If the earth's radius is 4000 miles, what is the velocity in miles per hour of a point on the equator, due to the earth's axial rotation? What is its angular velocity in degrees per hour? In radians per hour?*

4. *What lineal velocity and rate of angular change of the Lick Telescope (situated in latitude 37° 20′ 25″ North) is caused by the earth's axial rotation?*

5. *A fly wheel 10 feet in diameter makes 30 revolutions per minute. What is the lineal velocity of a point on the circumference?*

6. *What points on this wheel are moving at the rate of one mile per hour?*

7. *The moon is about 250,000 miles from the earth, and completes one revolution in about 28 days. What is its orbital velocity in miles per hour? What is its angular velocity in radians per hour? In degrees per hour?*

8. *The distance from the earth to the sun is about 92,000,000 miles. If there were exactly 365 days in a year, what would be the orbital velocity of the earth's center and its angular velocity?*

9. *Two gear wheels having diameters of 4 inches and 12 inches, respectively, have fixed axes O and O′. If the larger wheel has 200 revolutions per minute, how many revolutions does it impart to the smaller wheel? What is the lineal velocity of each point on the circumference of the small wheel, and what is its angular velocity about O? What is the lineal velocity of each point on the circumference of the larger wheel? Would the result have been the same if the motion had been transmitted from a 12-inch pulley to a 4-inch pulley by belt, instead of by gear?*

10. *A lathe is connected with a system of shafting, as illustrated in Fig. 23, the numbers indicating the diameters of the pulleys in inches.* The speed of the main shaft is 200 revolutions per minute.*

* In the diagram it will be noticed that the belt connecting the two cones passes from a 9½-inch to a 3-inch pulley, and that likewise the sum of the diameters of the two opposite pulleys in each step of the cones is always the same, 12½ inches, according to the figures in the diagram. This is not just as it should be. If the belt were made the right length for the "9½ ··· 3" inch combination, it would be a trifle too long for the other steps, particularly

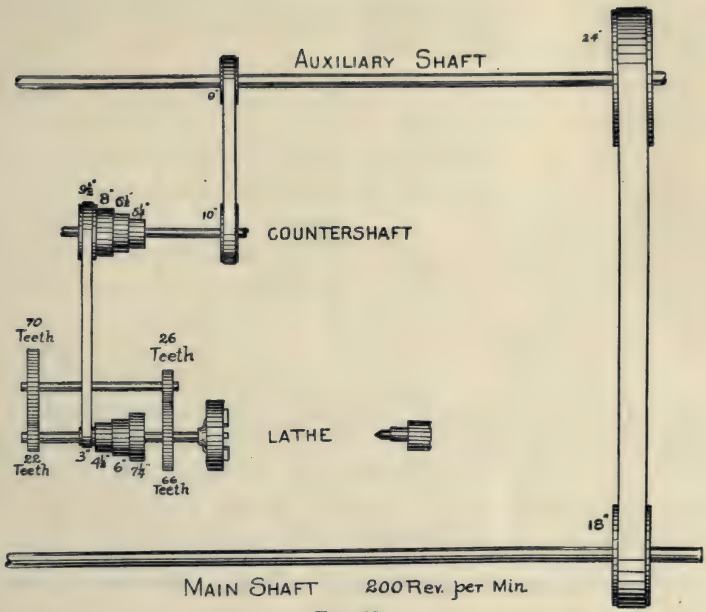


FIG. 23

(*a*) *Find the revolutions per minute, and the lineal velocity of the perimeter of each of the pulleys mentioned in the following table:*

PULLEY	ANGULAR VELOCITY Revolutions per Minute	LINEAL VELOCITY Feet per Minute
18 inch		
24 inch		
9 inch		
10 inch		
5½ inch		
6½ inch		
8 inch		
9½ inch		

for the pair that are most nearly equal. This should be allowed for and corrected in designing the cones.

(*b*) *Find the angular velocity of the chuck, with and without back gear. Tabulate results as follows:*

Position of Belt	Without Back Gear	With Back Gear	
	Angular Velocity Cone and Chuck	Angular Velocity of Back Gear	Angular Velocity of Chuck
On $7\frac{1}{4}$-inch Pulley			
On 6-inch Pulley			
On $4\frac{1}{2}$-inch Pulley			
On 3-inch Pulley			

11. *A seconds pendulum is one that requires two seconds for a complete oscillation forward and back, or double beat, and the length of a pendulum that beats seconds at sea level in middle latitudes is about 99.4 cm. If such a pendulum swings through an arc of 5 degrees, what are its average lineal and angular velocities?*

Composition of Circular and Rectilinear Motions. — When a ball is thrown in any direction it usually possesses a rotary or whirling motion, in addition to and independently of its progressive motion or motion of translation. It is by his ability to produce and control these rotations that a skillful baseball pitcher causes a baseball to move in an erratic path, — "pitching curves," as it is called. If the ball is moving in a straight line and rotating around an imaginary axis passing through its center, the actual motion, relatively to the ground, of any point on its surface at any instant is the resultant of a circular or rotational motion and a rectilinear motion or translation.

EXAMPLE

The Baseball Problem. — *A baseball, 2.75 inches in diameter, is thrown horizontally with a velocity of 80 feet per second, and at the same time is made to rotate at the rate of 30 revolutions per second around a horizontal axis at right angles to the direction in which it is thrown.*

In Fig. 24, let the circle ABCD represent a vertical section through the center of the ball. The line OX represents the horizontal velocity, and the curved arrow indicates a rotation around an axis imagined to pass through O, the center of the ball, perpendicular to the plane of the paper.

(a) Find the resultant velocity of each of the points A, B, C, and D, at the instant when they are in the position shown in the figure.

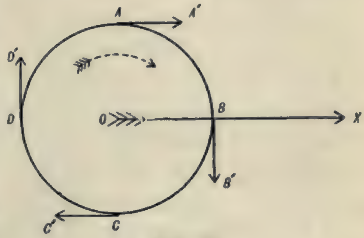


FIG. 24

DISCUSSION. — The entire mass of the ball, including the points A, B, C, and D, moves horizontally with the same velocity OX as the center, and hence OX is one of the component velocities for each of these points.

The other component, due to the rotation of the ball, is different for each point of the ball; for all points on the circumference of the circle $ABCD$, the *magnitude* of this rotational component is the same, but no two points are moving in the same *direction* at the given instant. When C is at the lowest position on the ball, the rotation gives it a motion for the instant in the direction of the tangential arrow CC'; the rotational component of point D at that instant would be vertically upward, of A horizontally to the right, and of B vertically downward.

It is important that the student should get a clear conception of what is meant by these tangential components. True, the points move in the circle and not along the tangential lines. They follow the circular path, however, merely because they are constrained to do so. It is not difficult to see, as the ball revolves and C moves from its present position to that now occupied by D, that if at the instant when it reaches the latter point it is freed from this constraint (which would pull it around in the circle toward A), it will move instead along the tangent DD', — like a drop of water similarly freed and flying from a grindstone.

While the points A, B, C, and D are not thus freed from the ball, still their respective motions are for the instant in the direc-

tions of the tangents. Imagine the circle made up of an infinite number of very small straight lines, — a regular polygon of an infinite number of sides, — and the same conclusion is reached. The infinitesimal line constituting any part of the circumference will have the direction of the tangent at that point. The rotational component of D is vertically upward for only an infinitesimal fraction of a second, and then it assumes a different direction. But, if we wish to combine this rotational component with the horizontal component OX, we must represent it graphically on the same scale. Hence, if OX represents one of the velocities of D in feet per second, then the other component DD' must also be represented in feet per second. The point D does not continue to move in direction DD' for a full second, but if it did it would cover the distance DD' on the same scale as OX. When we say that a train has a velocity of 30 miles per hour, we do not insist that it shall move the full hour, but we can represent its motion graphically as if it did; so we say of the ball, if point D moves a very small distance in direction DD' in a very small interval of time, at the same rate it would in a full second move the distance DD'.

Hence, the motion of D, relatively to the ground at the given instant, is the resultant of the two components OX and DD' at right angles to each other. And this will determine the direction of the resultant as well as its magnitude. The two components of B are also at right angles to each other; those of A and C are parallel.

Notice that the ball is represented in Fig. 24 on a scale much larger than the scale of velocities. If it makes 30 rotations per second it thereby causes the points A, B, C, and D to move in a second through a distance equal to 30 times the circumference shown in the figure, which would be many times greater than the line DD'. It is obvious that the velocities had to be represented on a greatly reduced scale.

(*b*) *Find the magnitude and direction of the resultant velocity of a point on the ball between A and B, 22° from B.*

The motion of a carriage wheel is not unlike that of the ball in the preceding example, except that the rate of rota-

tion of the wheel is not independent of the velocity of the vehicle. A baseball may revolve fast or slowly, without regard to the velocity with which it is thrown, but the rate at which a carriage wheel revolves necessarily bears a fixed relation to the velocity of the vehicle. In one revolution the wheel measures the length of its circumference on the ground. Hence, the magnitude of the rotational component of any point on the perimeter of the wheel is exactly equal to the rectilinear component parallel to the ground. Otherwise, the general conditions are the same as those of the baseball in the preceding example.

EXAMPLES

1. *We know that that point on the tire of a wheel momentarily touching the ground is at rest relatively to the ground;* otherwise there would be slipping. Prove this by composition of the rectilinear and rotational components.*

2. *If* v *is the velocity of the vehicle, prove that the highest point on the wheel has a resultant velocity of* 2 v *in the same direction.*

3. *A carriage is traveling at the rate of 9 miles per hour.*

(*a*) *Find the resultant velocity relatively to the ground of a point 40° from the ground on the front side of one of the wheels.*

(*b*) *If the diameter of the front wheels is $3\frac{1}{2}$ feet and of the hind wheels 4 feet, find the angular velocity of each.*

Composition of two Circular Motions. — Every point on the earth's surface is subject to at least two motions, one due to the rotation of the earth on its axis, and the other due to its orbital motion or revolution around the sun. Hence, the resultant motion of a point A (Fig. 25) is different from that of a point B.

* If there is any doubt of this in the mind of the student, try the experiment of tying a piece of white cloth, or marking with a piece of chalk, around a bicycle tire. It will be seen that when the bicycle is ridden, the cloth or chalk mark comes to rest as it reaches the ground.

As the moon revolves around the earth, however, it does not possess an additional *independent* motion around its own

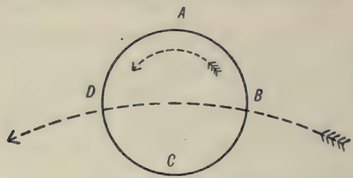


FIG. 25

axis. Its motion in this respect is analogous to the motion of a circular disk fastened to a stick, as in Fig. 26. If this entire device is made to revolve around point O of the stick in the direction of the arrow, a person standing beyond A would see successively points A, B, C, D. To him, therefore, there would be an apparent rotation of the disk around its center, but this apparent rotation is really controlled by the revolution about O.

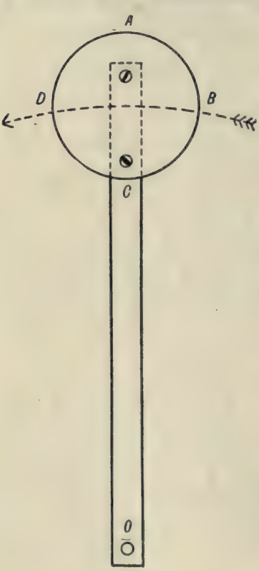


FIG. 26

This does not include the consideration that the moon is carried with the earth in the motion of the latter around the sun, and hence has a double motion after all. In fact, if the moon had an independent axial rotation, each point in it would then be subject to three simultaneous curvilinear motions.

Since the earth's axis is inclined to its orbit, and since the moon's orbit around the earth is inclined to the earth's orbit, the determination of the resultants of these motions is not simple enough for our purposes of explanation. The same ideas, however, are illustrated in modified form in the examples following.

EXAMPLES

1. *A circular disk 6 inches in diameter is fastened to a stick. This device is made to revolve 25 times per minute around an axis which passes through the stick at a point 18 inches from the center of the disk. Find the tangential velocity of the center of the disk and of each of four points situated as A, B, C, and D, Fig. 26 (line BD being perpendicular to AC through the center of the circle ABCD).*

2. *Imagine the screws removed and the disk no longer fixed rigidly to the stick, but free to rotate around a pin P at its center, as in Fig. 27. If the disk rotates 70 times per minute around P, in addition to its revolution around O, what will be the resultant velocity of each of the points P, A, B, and C?*

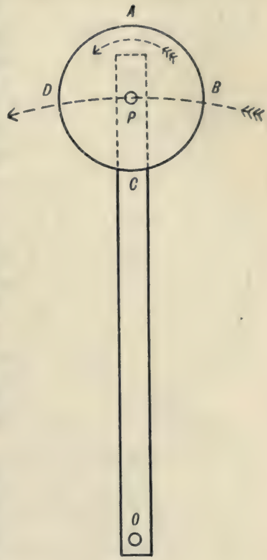


Fig. 27

This is an arrangement that would be quite analogous to the axial and orbital motions of the earth, if the earth's axis were perpendicular to the plane of its orbit.

Comparing this example with the baseball problem will show how the composition of two circular motions differs from the composition of a circular with a rectilinear motion. The revolution of the disk (Fig. 27) in a circular path around *O* makes one of the components of point *A* greater than the corresponding component of point *C*, in proportion as the radius *OA* is greater than the radius *OC*; but if the disk were moving in a straight line, like the baseball in Fig. 24, this rectilinear component would be the same for all points.

In the baseball problem every point on the perimeter of the section shown in the figure had the same pair of components, the only difference for different points being in the angle between the two components; in Fig. 27, not only is the angle different for each pair of components, but one of the components is also different.

3. *If the earth's axis were perpendicular to the plane of its orbit, what would be the resultant velocity of a point on the equator at midnight, at noonday, and at 7 o'clock P.M.?*

Assume: One year equal to 365 days; diameter of the earth at the equator equal to 8000 miles; distance of the center of the earth from the sun equal to 92,000,000 miles; that the earth revolves around the sun from west to east and has its axial rotation in the same direction.

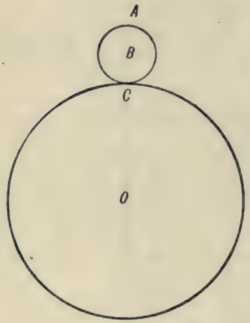


FIG. 28

4. *If the moon's orbit around the earth were in the same plane as the earth's orbit, what would be the resultant velocity of the center of the moon at "new moon"?*

Assume that the moon completes its circuit around the earth in 28 days, and that the motion is from west to east, like the orbital motion of the earth.

5. *A circle 6 inches in diameter rolls around a larger circle, diameter 24 inches, at the rate of 50 revolutions per minute. B is the center of the small circle, and A and C points on its circumference. What are the lineal and angular velocities of B? What is the resultant lineal velocity of A and of C?*

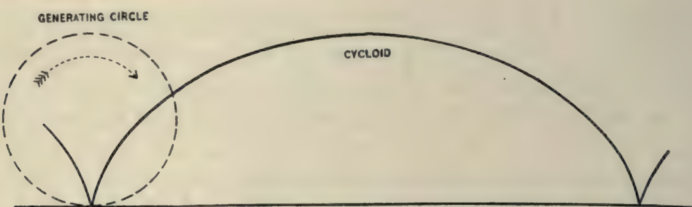


FIG. 29

This problem bears the same relation to the three examples preceding that the carriage-wheel problem bears to the baseball problem; the axial rotation of the earth is not in a simple ratio to its orbital motion, being entirely independent of it, but in this problem the two motions of the small circle are interdependent, if there is no slipping.

Cycloids. — If a piece of chalk or other marker be fastened at the perimeter of a disk, and the latter be rolled along the floor with the disk against the wall, the marker will trace a

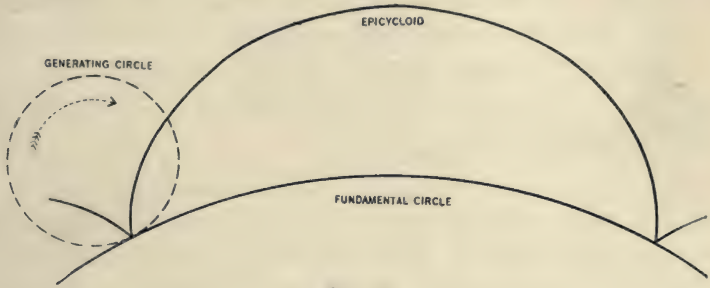


FIG. 30

line like the curve shown in Fig. 29. This curve is called a **cycloid**.

Every point on a carriage wheel describes such a path. An inspection of this curve discloses the fact, already referred to, that every point on the perimeter of the disk or wheel comes to rest as it reaches the ground. Approaching the ground, it is moving almost vertically downward, and the instant it

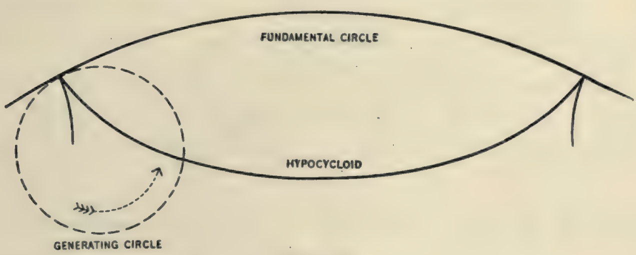


FIG. 31

leaves the ground its motion is upward. Having turned back abruptly, as the diagram shows, it must have come to rest at the turning point.

The cycloid generated by a point on the circumference of a circle as it rolls along a straight line is only one of a class

of similar curves that are used in mechanics, especially in designing gear teeth. If the circle, instead of rolling along a straight line, be rolled on the circumference of another circle, as in Fig. 30, the curve generated is called an **epicycloid**. If it be rolled on the inside of a ring in the same direction as before, its rotations around its own axis will be in the opposite direction, and the curve generated in this case is called a **hypocycloid**.

CHAPTER IV

ACCELERATION

A PERFECTLY uniform rate of motion is probably never realized. The movements of a clock or of a rotating shaft are generally assumed to be uniform, but are really subject to many fluctuations. An electric car moves with increasing velocity up to the desired speed, maintains an apparently uniform velocity for a time, and again assumes a distinctly variable velocity before stopping. A falling body gains velocity continuously; and if hurled upward, loses velocity steadily until it again starts downward.

The subject of **acceleration** deals with all cases of variable velocity. When the rate of motion changes irregularly or spasmodically, it generally involves too many complexities for treatment by arithmetical or algebraic methods. Hence, our study of acceleration will be limited to motions that change uniformly, — called uniformly accelerated motions.

It should be remembered that changes of direction, as exemplified by motion in a circle, need not affect the rate of a motion. Unless otherwise specified, a motion is always assumed to be in a straight line.

Suppose that a body is observed at a given instant to have a velocity of 10 miles per hour, represented by the line *AB*, Fig. 32. If, after a few minutes, say three minutes, it is found to have a velocity of 16 miles per hour, represented by the line *CD*, the question arises: Was this

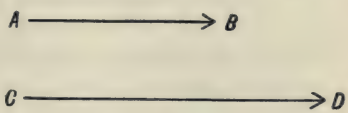


FIG. 32

additional velocity caused by some sudden impulse, or was it a steady increase distributed equally over the entire interval of three minutes? If we take the latter supposition, and divide the total increase of velocity (6 miles per hour) by the time during which the increase took place (3 minutes), the quotient will represent the amount of velocity (in miles per hour) by which the motion of the body was increased during each of the three minutes, or its acceleration in miles per hour, per minute. At the end of the first minute the velocity must have been 12 miles per hour; at the end of the second minute, 14 miles per hour, etc. At the end of $1\frac{1}{2}$ minutes it was 13 miles per hour; at the end of $2\frac{1}{4}$ minutes

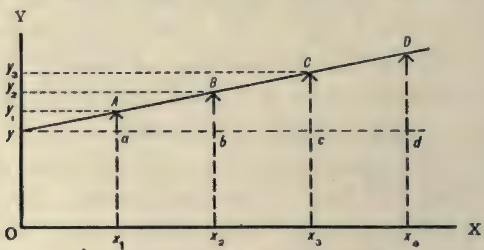


FIG. 33

it was $14\frac{1}{2}$ miles per hour, etc. The rate of motion is not the same at any two successive instants, and the body does not travel through any appreciable distance at a fixed rate. The velocity at any instant is the distance the body would move over in the next unit of time, if it continued to move uniformly for that unit of time, at the same rate as at the instant in question.

Much of this can be shown graphically. Draw a horizontal base line OX and a vertical base line OY. Let intervals of time be represented on a convenient horizontal scale, say 1 inch = 1 minute, and use a vertical arrow to indicate the velocity at any instant, on a scale of 1 inch = velocity of 10 miles per hour. (The dimensions of Fig. 33 are half this scale.) Oy will represent the velocity as first observed,

and Oy_3 will represent the velocity as observed after the lapse of 3 minutes. Ox_3 will represent the lapse of 3 minutes. The line x_3C, parallel to OY, contains all points 3 inches from the Y-axis; the line y_3C, parallel to OX, contains all points 1.6 inches from the X-axis. The intersection of x_3C and y_3C locates a point which pictures the two things, viz. that at the end of 3 minutes (Ox_3) the velocity of the body is 16 miles per hour (x_3C). To show that the rate of motion has increased uniformly, we have only to connect y and C by a straight line. The upward inclination of yC shows that the rate of motion is increasing, and the steepness of the gradient pictures the rapidity of the change. The total increase of velocity for the 3 minutes is cC.

Supposing that the rate of motion of this body continues to increase at the same rate as observed for the interval of 3 minutes, can we calculate the velocity it would have at any subsequent time? If the rate of increase — 2 miles per hour, per minute — continues, then the velocity at the end of the fourth minute from the beginning will be $10 + 4 \times 2$, or 18 miles per hour, which would be represented by the line x_4D in the diagram.

EXAMPLES

1. *Taking a point x_1 (Fig. 33) so that Ox_1 will represent one minute, what will be represented by the perpendicular x_1A? What will aA represent? How could we find from the diagram the velocity of the body at the end of 1.7 minutes? At what time after starting would the velocity be 15.3 miles per hour? What does dD represent?*

2. *A body starting from rest has its velocity increased gradually and uniformly until at the end of one second its velocity is found to be 10 feet per second.*

(*a*) *What has been its average velocity meanwhile?*

(*b*) *Suppose its increase of velocity to be always at the same rate; what would it be at the end of the second second?*

(*c*) *What will have been its average velocity during the first two seconds?*

(*d*) *What during the second second?*
(*e*) *What during the first three seconds?*
(*f*) *What during the third second?*

3. (*a*) *What distance will be passed over by this body during the first second?*
(*b*) *During the first two seconds?*
(*c*) *During the second second?*
(*d*) *During the first three seconds?*
(*e*) *During the third second?*

4. *A train of cars leaving a station acquires velocity gradually and uniformly until at the end of three minutes it is moving at the rate of 30 miles per hour.*

(*a*) *What velocity in miles per hour has been added each minute?*

(*b*) *What velocity in feet per minute has been added each minute?*

(*c*) *What velocity in feet per minute has been added each second?*

(*d*) *What velocity in feet per second has been added each second?*

5. *A body starting from rest is accelerated uniformly. At the end of 5 seconds it has a velocity of 100 feet per second.*
(*a*) *What has been its average velocity during the five seconds?*
(*b*) *How far has it traveled?*
(*c*) *What was its velocity at the end of the first second?*
(*d*) *What is its acceleration?*
(*e*) *How far did it travel during the first second?*
(*f*) *What was its velocity at the end of the second second?*
(*g*) *How far did it travel during the first two seconds?*
(*h*) *How far did it travel during the second second?*
(*i*) *How far did it travel during the first ten seconds?*
(*j*) *How far would it travel during the fourteenth second?*

Definition of Acceleration.—By the acceleration of a body we mean the rate at which its rate of motion, or velocity, changes.

In the first illustration of this idea at the beginning of this chapter, the velocity changed from 10 miles per hour to 16 miles per hour in 3 minutes — the change taking place at the rate of 2 miles per hour for each minute; and hence, according to the definition, the acceleration was 2 miles per hour, per minute.

In Example 2, page 49, the acceleration was 10 feet per second, per second; in Example 4 it was 10 miles per hour, per minute, or $\frac{1}{6}$ mile per hour, per second (and might have been expressed in any other units of time and distance, as feet per minute, per minute; feet per minute, per second; feet per second, per second; inches per second, per second, etc.).

Formulæ for Uniformly Accelerated Motion. — From the preceding examples and statements we can generalize to the following relations:

(i) If a body starting from rest is subject to a uniform acceleration α, its velocity v at the end of any time t will be

$$v = \alpha t. \tag{4}$$

(ii) If the body does not start from a condition of rest, its motion may be either accelerated or retarded, the latter being regarded as a negative acceleration. If its initial velocity is v_1, the velocity v at the end of any given interval t will be

$$v = v_1 + \alpha t. \tag{5}$$

(iii) To find the distance traveled in any time by any body, no matter how it is moving, we multiply the average velocity by the time. If the body starts from rest and is subject to a uniform acceleration α, the velocity at the *end* of time t is αt; the average velocity *during* t is $\alpha t/2$. The distance traveled is

$$d = \frac{\alpha t}{2} \times t = \frac{\alpha t^2}{2}. \tag{6}$$

If, instead of starting from rest, the motion changes uniformly from an initial velocity v_1 to a final velocity v_2, then the distance traveled meanwhile will be

$$d = \frac{v_1 + v_2}{2} \times t.$$

But v_2 can be expressed in terms of v_1 and the acceleration, thus:

$$v_2 = v_1 + at.$$

Whence $d = \dfrac{v_1 + v_1 + at}{2} \times t = \left(\dfrac{2v_1 + at}{2}\right) t = v_1 t + \dfrac{at^2}{2}$. (7)

Galileo was the first to explain with mathematical exactness the idea of accelerated motion. In 1638 he made known the relations shown in Formulæ 4 and 6.

EXAMPLES

1. *A body starting from rest moves with a uniformly accelerated motion. When it has passed over a distance of 100 feet its velocity is found to be 50 feet per second.*

(*a*) *What has been its average velocity meanwhile?*

(*b*) *How long has it taken the body to travel the 100 feet?*

(*c*) *What is its acceleration?*

(*d*) *What was its velocity at the end of 2.6 seconds?*

(*e*) *What will be its velocity at the end of one minute, and how far will it have traveled meanwhile?*

2. *A body starting with a velocity of 100 feet per second loses its velocity gradually and uniformly and comes to rest in 6 seconds.*

(*a*) *What is its acceleration?*

(*b*) *How far will it have traveled before coming to rest?*

(*c*) *How far will it have traveled at the end of 3 seconds?*

3. *A body starting with a velocity of 30 feet per second is accelerated uniformly at the rate of 8 feet per second, per second.*

(*a*) *What will be its velocity at the end of 5 seconds?*

(*b*) *How far will it have traveled meanwhile?*

(*c*) *How far will it travel during the thirteenth second?*

4. *A body starting with a velocity of 50 feet per second is accelerated uniformly. At the end of 5 seconds it has a velocity of 200 feet per second.*

(*a*) *What is its acceleration?*

(*b*) *How far does it travel during the second, third, and fourth seconds, inclusive?*

5. *A body starting with a velocity of 50 feet per second is accelerated uniformly. At the end of 10 seconds it is found to have traveled 2000 feet.*

(*a*) *What has been its average velocity?*

(*b*) *What was its final velocity?*

(*c*) *What was its acceleration?*

6. *A body starting with a velocity of 50 feet per second loses velocity at a uniform rate. At the end of 3 seconds it has a velocity of 20 feet per second.*

(*a*) *What was its acceleration?*

(*b*) *How long before it will come to rest?*

(*c*) *How far will it move before coming to rest?*

7. *If a body starts from rest and moves through the distance* d *while it is acquiring a velocity* v, *prove that*

$$\mathrm{d} = \frac{\mathrm{v}^2}{2\,a}, \qquad (8)$$

where a is the acceleration.

This relation (Formula 8) was not mentioned by Galileo when he enunciated the laws expressed by Formulæ 4 and 6, page 51, but was worked out afterward by Huygens.

Acceleration of Gravity. — The acceleration of falling bodies at sea level is about 32.2 feet per second, per second, while bodies hurled upward lose velocity at this same rate. This acceleration is usually represented by g, although a more precise symbol would be one that signifies acceleration due to weight rather than gravity, because there is a difference between the observed value of g at a given place and the true value of gravitational attraction. The latter is dimin-

ished in effect by the earth's rotation, in such manner that the apparent value of g is greatest at the poles and least at the equator. In this book 32.2 will be used as an approximate value of g for all places, unless otherwise specified. The expression "acceleration of gravity" will also be used instead of "acceleration due to weight," with the understanding that "gravitational attraction" when applied to a specific place on the earth's surface is subject to correction.

EXAMPLES

1. *A body falling from a certain height requires 5 seconds to reach the ground. With what velocity does it strike? What was its velocity at the end of 1.5 seconds? At the end of 3.75 seconds? From what height was it dropped? What part of the distance was covered in each of the five seconds? What part was covered in the first 1.7 seconds? In the first 4.4 seconds? During the interval between the end of the second second and the end of 3.8 seconds?*

2. *A falling body strikes the ground with a velocity of 193.2 feet per second. From what height was it dropped?*

3. *A body is dropped from a height of 788.9 feet. How long before it will strike the ground? With what velocity will it strike?*

4. *A body is hurled downward with a velocity of 100 feet per second. What will be its velocity at the end of 3 seconds? How far will it have traveled? How far will it have traveled at the end of 5 seconds? What part of this distance will have been covered during each of the 5 seconds? During second and third seconds, inclusive?*

5. *A body having been hurled vertically downward strikes the ground at the end of 7 seconds with a velocity of 525.4 feet per second. With what velocity was it hurled? From what height was it hurled?*

6. *A body is hurled downward with a velocity of 20 feet per second from a height of 1810 feet. How long before it will strike the ground?*

7. *A body is projected vertically upward with a velocity of 1000 feet per second. How long will it continue to rise? How far will it rise? With what velocity will it strike the ground on returning?*

8. *A man reaching from the top of a tower 644 feet high, throws a ball vertically upward with a velocity of 96.6 feet per second. How long before it will reach the ground?*

9. *If the acceleration of gravity is* g *feet per second, per second, express the following relations:*

(i) v *in terms of* g *and* t;
(ii) h *in terms of* g *and* t;
(iii) h *in terms of* g *and* v.

The body is assumed to start from rest; v *is its velocity at the end of* t *seconds, and* h *is the distance it falls meanwhile.*

10. *What is the value of* g *expressed in centimeters per second, per second?*

11. *A body thrown vertically upward rises to a height of 40 feet. With what velocity was it thrown, and how long was it in ascent?*

Use of Coördinate Axes. — The method of locating points with reference to two lines or axes, employed in Fig. 33, page 48, and elsewhere in this book, is used very widely in various branches of mathematics and science. The two reference lines may be drawn at any angle with each other, although they are usually perpendicular, one being drawn horizontally and the other vertically. Their intersection is called the **origin**. The horizontal line is called the **X-axis**, or axis of abscissæ, and the vertical line is the **Y-axis**, or axis of ordinates.

The familiar geographical method of locating places by means of latitude and longitude assumes two such reference lines, — the terrestrial equator and the arbitrary line through Greenwich called the zero meridian. What do we mean, for example, when we say that a place is situated in latitude 30° N., longitude 80° W.? The thirtieth parallel of north latitude is the location or

locus of all points 30° north of the equator, and the eightieth meridian west is the *locus* of all points 80° west of the zero or Greenwich meridian. The intersection of these two lines or *loci* is the point designated.

In Fig. 33, page 48, by locating points A, B, C, etc., with reference to two axes at right angles to each other, we represented the gradual changing of a velocity during several seconds, in accordance with the law expressed in Formula 5, page 52, $v = v_1 + at$. Now it is true generally that any relation between two quantities can be thus represented as a line, either straight or curved, and the character of this line will picture the nature of the relation between the two quantities.

With reference to two such axes every simple proportion and every relation between two quantities shown by an equation of the first degree can be represented by a straight line. The weight of a given volume of anything depends upon its density, or $W = V \times d$, in which V is a constant and W is a function of d; the electrical resistance of a wire is proportional to its length; the bending of a beam is directly proportional to the load upon it. Each of these relations could be represented by a straight line.

More complex relations give rise to curves of various sorts or even to broken lines. Some quadratic equations produce ellipses, others circles, parabolas, etc. Temperature curves and barometric curves, showing the gradual rise and fall of the thermometer or barometer throughout a day or other interval of time, are usually

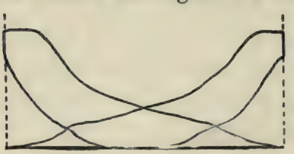


FIG. 34

very irregular. A very interesting case is the diagram on an indicator card from which the horse power of an engine is determined, the relation in this case being between the pressure of steam in the cylinder and the distance of the piston from either end of the stroke. Figure 34 shows such a diagram * taken from an engine of 12-inch stroke

* Taken from a 9.5-inch x 12-inch Armington & Sims Standard Automatic Cut-off Engine running 200 revolutions per minute, under boiler pressure of 75 pounds per square inch, indicating 20.26 H.P.

under a boiler pressure of 75 pounds per square inch, so that the total horizontal width of the diagram corresponds to 12 inches of stroke and the highest point on the curves corresponds to about 75 pounds pressure per square inch.

In every case, whether straight or curved, the line pictures a relation between two quantities: velocity and time; temperature and time; weight and density; etc., one of the quantities being a function of the other, or the changes in one being accompanied by changes in the other.

Graphical Representation of Distance traveled by an Accelerated Body. — To the lines and points in a diagram constructed upon coördinate axes various meanings may be attached, depending

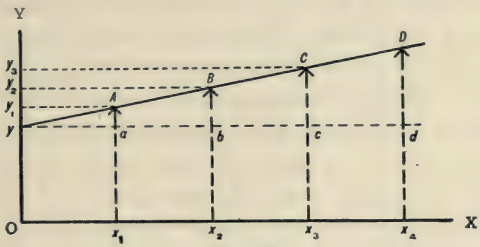


FIG. 35

upon the idea to be illustrated. Sometimes, furthermore, the usefulness of the figure depends upon the interpretation of the area included within the various lines and axes as much as upon the lines themselves. For example, in the indicator diagram from which the power and workings of a steam engine are investigated, the horse power is determined from the area within the curves.

Reverting to Fig. 33, page 48, which is here reprinted as Fig. 35, it can be shown that the area Ox_3Cy represents the distance traveled by the body referred to. Oy represents the initial velocity, and Ox_3 represents an interval of time — 3 minutes. If the velocity had continued equal to Oy (10 miles per hour) throughout the 3 minutes, — that is, if there had been no acceleration, — the distance traveled during that time would have been $v \times t = \frac{10 \times 3}{60} = \frac{1}{2}$ mile. In the diagram v is represented

by Oy and t is represented by Ox_3. The product of Oy and Ox_3 is the area of the parallelogram Ox_3cy, whence it appears that the distance traveled by any body moving with a uniform velocity can be represented by the area of a rectangle of which two adjacent sides represent the velocity and time, respectively.

The same method applies to any motion. If the body under consideration, instead of moving uniformly at the rate of 10 miles per hour has an acceleration of 2 miles per hour, per minute, increasing from Oy in the diagram to Oy_3 or x_3C in the course of 3 minutes, it is apparent that the area Ox_3cy will represent only a part of the total distance. Study the diagram by comparison with the formula $d = v_1t + \frac{\alpha t^2}{2}$. It has just been shown that the term v_1t is represented by the area Ox_3cy, which indicates what the distance would have been if the velocity had remained constant. The term $\alpha t^2/2$ is represented by the area of the triangle ycC,* and indicates the additional distance covered on account of the increased motion. The total area Ox_3Cy, therefore, represents the sum of the two, or the distance traveled in 3 minutes by a body starting with a velocity of 10 miles an hour and subject to an acceleration of 2 miles per hour, per minute.

This method could be used even for variable accelerations, thereby avoiding many complex computations. No matter how erratic the changes from velocity Oy to velocity x_3C, if they can be put into a diagram as in Fig. 36, we can easily determine the area Oxy_1y, which represents the distance traveled. This area, divided by the time Ox, will give the average velocity or the average of all the possible vertical coördinates.

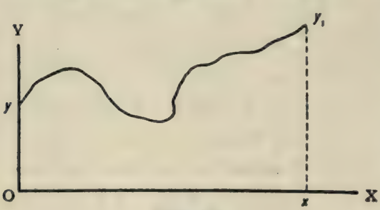


Fig. 36

[A simple plan for determining the irregular area Oxy_1y would

* $cC = \alpha t$ and $yc = t$; whence the area of the triangle is equal to $\frac{yc \times cC}{2}$, or $\frac{\alpha t^2}{2}$, as stated.

be to cut out the figure from a piece of cardboard and compare the weight of this irregular piece of cardboard with the weight of a regular piece, or known area, of the same material.]

PROJECTILES

The path of a free projectile, undisturbed except by the action of gravity, is always a curve. Whatever the velocity or angle of projection, this curve always possesses certain characteristic features or mathematical relations peculiar to a class of curves called **parabolas.** A projectile is always subject to two component motions, one of which (the velocity of projection) is constant in both magnitude and direction, while the other (the gravitational component) is constant in direction but of variable magnitude. If both components were entirely uniform velocities, the resultant path would be a straight line instead of a curve, as in Figs. 3 and 4, pages 18 and 19. If it were not for the action of gravity, a free and unobstructed body hurled in any direction — horizontally, obliquely, or vertically — would continue to move in a straight line with a uniform velocity.* The effect of gravitation, however, adds this variable downward component, the value of which at any instant can be determined by the law of falling bodies. The motion of a projectile at any instant is, therefore, the resultant of two components, — one a uniform velocity in the direction of projection, and the other a uniformly accelerated velocity vertically downward.

For instance, suppose a ball is rolled along a platform OA, Fig. 37, with a uniform velocity, say 40 feet per second. If the platform remains at rest, the position of the ball at the end of successive seconds will be at points 1, 2, 3, etc. But if the platform is allowed to fall freely, at the end of 1 second it will be in position O_1A_1, 16.1 feet below OA, and the ball will be at 1′. At the end of 2 seconds the ball will be at 2′, 64.4 feet below 2, etc.

* See p. 22.

It should be noted that this diagram (Fig. 37) is very different from some of the graphical representations previously used; in this case the horizontal and vertical lines represent distances and not velocities, and the curved line is not the resultant velocity, but the actual path traversed. The resultant velocity at any instant is a tangent to the curve at the point where the body happens to be at that instant. To compute this resultant we may combine the fixed horizontal component and the computed vertical component for the given instant. These velocities may be represented graphically on any scale desired, independently of the scale of distances used in constructing the path. At point O, for example, the horizontal component is 40 feet per second and the vertical component is zero; the resultant is 40 feet per second horizontally. At the very next instant — the merest fraction of a second — it is something different. At point $1'$ the horizontal component is 40 feet per second and the vertical component is 32.2 feet per second; the resultant is $\sqrt{40^2 + 32.2^2}$, at an angle $\delta = \tan^{-1} 32.2 \div 40$.* At point $3'$ the horizontal component is still 40 feet per second and the vertical component is 96.6 feet per second; the resultant is $\sqrt{40^2 + 96.6^2}$, at an angle $\delta = \tan^{-1} 96.6 \div 40$.

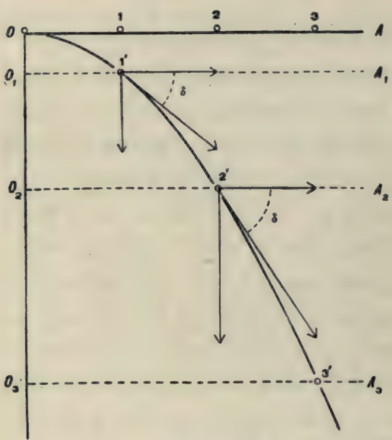


FIG. 37

The resultant of two uniform components is constant in both magnitude and direction; uniform motion in a circle is constant

* $\tan^{-1} 32.2 \div 40$ means "the angle whose tangent is $32.2 \div 40$."

in rate, but continually changes direction; the motion of a projectile changes in both magnitude and direction.

EXAMPLES

1. (*a*) *Construct a diagram of the path of a projectile hurled horizontally with a velocity of 100 feet per second from a height of 579.6 feet.*

Instructions. — For graphical representation, use a scale of 1 inch = 100 feet. First verify the table of distances printed below, in which the time refers to the number of seconds after starting and the corresponding horizontal and vertical distances traveled are reduced to scale.

Prepare a cross-section sheet divided into one-inch squares with extra vertical lines at half-inch intervals where required by table. Beginning at upper left-hand corner as the origin, locate points determined by values reduced to scale in table, and connect successive points, starting from origin. In preparing diagram use a sharp, hard pencil.

Time	Horizontal Distance [Reduced to Scale]		Vertical Distance [Reduced to Scale]	
$t = 0.5$ seconds	50 feet	0.5 inches	4.0 feet	0.040 inches
1.0 seconds	100 feet	1.0 inches	16.1 feet	0.161 inches
1.5 seconds	150 feet	1.5 inches	32.2 feet	0.322 inches
2.0 seconds	200 feet	2.0 inches	64.4 feet	0.644 inches
2.5 seconds	250 feet	2.5 inches	100.6 feet	1.006 inches
3.0 seconds	300 feet	3.0 inches	144.9 feet	1.449 inches
4.0 seconds	400 feet	4.0 inches	257.6 feet	2.576 inches
5.0 seconds	500 feet	5.0 inches	402.5 feet	4.025 inches
6.0 seconds	600 feet	6.0 inches	579.6 feet	5.796 inches

(*b*) *Compute the resultant velocity of this body at the instant of striking the ground, giving the direction of the resultant as well as its magnitude.*

(*c*) *What was its resultant velocity at the end of the third second?*

2. *A body is projected with a velocity of 20 feet per second at an angle of 35° above the horizon. Construct a diagram of its path.*

INSTRUCTIONS.— Use a scale of 1 inch = 5 feet. Tabulate values of distances traveled in direction of projection, and the falling distances, for successive periods of time changing by $\frac{1}{10}$ second, after manner of last example. Trace the path up to the end of 1.5 seconds. In ruling the cross-section sheet, draw the usual vertical base line, but construct the second reference line at the proper angle of 35° above the horizon. On the latter, measure off distances of 0.4 inch (representing distance of 2 feet traveled in that direction during each 0.1 second), and through each of these points draw a vertical line. In locating points determined by the values in table, count the uniform values along the inclined reference line, and from each of the successive points marked on that line measure the falling distance for the time interval referred to. The path should resemble the curve in Fig. 38.

3. *Prove:*

(*i*) *A body hurled horizontally from any altitude reaches the ground in the same time as if it had been merely dropped and allowed to fall vertically.*

(*ii*) *A body projected at any angle reaches the ground in the same time as if it had been projected vertically with a velocity equal to the vertical component of the actual velocity of projection.*

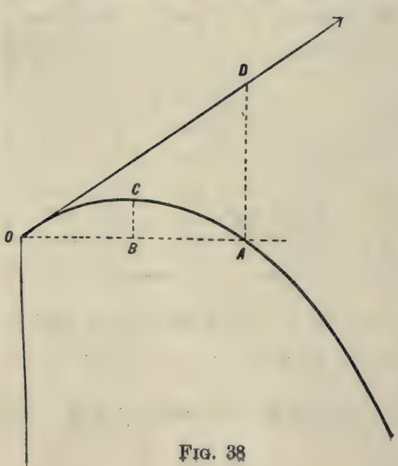


FIG. 38

Elevation and Range of a Projectile. — The angle of projection above the horizon is called the **elevation.** The horizontal distance OA, Fig. 38, is called the **range.** The highest point of the path is called its **vertex,** of which BC is the height.

The range and height

of vertex are easily determined by computation. For this purpose resolve the velocity of projection into its horizontal and vertical components. A body projected with a velocity v_p, at an angle β, will rise to the same vertical height and in the same time as if it were thrown vertically upward with a velocity equal to component a. Now we have already learned how to find the height to which a body will rise if hurled upward with a given velocity, and the time that will elapse before it returns to the ground. This time multiplied by the horizontal component will give the range.

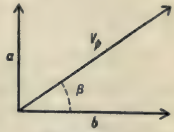


FIG. 38 *a*

For example, the range and height of vertex in Example 2, page 62, would be determined as follows:

$$v_p = 20$$
$$a = v_p \sin 35° = 20 \times 0.5736 = 11.47$$
$$b = v_p \cos 35° = 20 \times 0.8192 = 16.38.$$

An upward vertical velocity suffers an acceleration of -32.2 feet per second, per second, on account of gravity, whence this body will reach its highest point in

$$t = \frac{11.47}{32.2} = 0.356 \text{ second},$$

at an altitude of

$$h = \frac{11.47}{2} \times 0.356 = 2.04 \text{ feet}.$$

To find the range R, multiply the horizontal component of v_p by $2t$, the time that elapses while the body moves to the highest point and back to the horizontal.

$$R = 2 \times 0.356 \times 16.38 = 11.66 \text{ feet}.$$

EXAMPLES

1. *From your diagram constructed for Example 2, page 62, determine the range and height of vertex by scale, and compare with the above results.*

2. *A body is projected with a velocity of 20 feet per second at an elevation of 45°. Construct a diagram of its path up to the end of 1.5 seconds. Determine its height of vertex and range by scale and by computation.*

3. *Do likewise for a projectile hurled with a velocity of 21 miles per hour at an elevation of 55°.*

4. *A shell is fired with a velocity of 30,000 feet per minute at an elevation of 47°. If the gun is 400 feet above sea level, compute the time at which the shell will strike the water, the horizontal distance from the bottom of the cliff to the striking point, and the greatest altitude reached.*

5. *A 20-pound weight is dropped from a window of a car traveling over a bridge at the rate of 30 miles an hour. How long will it take to reach the water 75 feet below?*

Maximum Range of a Projectile. — If it were not for the resistance of the air, a projectile hurled with any velocity would have a maximum range when the angle of elevation is 45°. In Fig. 38,

$$OA = OD \cos \beta,$$

where β is the angle of elevation. But

$$OD = v_p t_a,$$

in which t_a is the time to reach A, or the time that elapses while the body is moving to the highest point and back to the horizontal. And

$$t_a = 2\frac{v_p \sin \beta}{32.2},$$

whence $$OA = v_p t_a \cos \beta = \frac{2\, v_p^2 \sin \beta \cos \beta}{32.2}.$$

But $$2 \sin \beta \cos \beta = \sin 2\, \beta,$$

whence $$OA = \frac{v_p^2}{32.2} \sin 2\, \beta.$$

Since $v_p^2 \div 32.2$ is a constant, the range OA is a function of $\sin 2\, \beta$, and hence OA has its maximum value when

sin 2β is greatest. The greatest value that the sine of any angle can have is the sine of 90°, or *one*. Putting $\sin 2\beta = 1$, 2β will then be 90°, or $\beta = 45°$. Therefore, the range OA is greatest when $\beta = 45°$, whatever the velocity of projection.

Striking Velocity Independent of Angle of Elevation. — For a given velocity of projection the velocity with which the body strikes is independent of the angle of elevation. This refers to the magnitude of the striking velocity and not to its direction. The larger the angle of elevation the longer the body remains in flight, but when it strikes it has the same velocity whether it was projected vertically upward, or vertically downward, or at any angle between. If we assume any height h_1 above or below the plane of OA in Fig. 38, it can be shown that the resultant velocity of the projectile when it reaches the new level is equal to $\sqrt{v_p^2 + 2\,gh_1}$, and hence independent of β.

Projecting the body with a velocity v_p at an angle β is equivalent to projecting it with a horizontal velocity $v_p \cos\beta$ and a simultaneous vertical velocity $v_p \sin\beta$. If h is the height of vertex in Fig. 38, then

$$h = \frac{(v_p \sin\beta)^2}{2\,g}.$$

When the body reaches a level h_1 below OA, the vertical component of its velocity will therefore be

$$\sqrt{2\,g(h_1 + h)} \text{ or } \sqrt{2\,g\left(h_1 + \frac{v_p^2 \sin^2\beta}{2\,g}\right)} \text{ or } \sqrt{2\,gh_1 + v_p^2 \sin^2\beta}.$$

The horizontal component remains $v_p \cos\beta$, whence the resultant velocity of the body is

$$\sqrt{2\,gh_1 + v_p^2 \sin^2\beta + v_p^2 \cos^2\beta} \text{ or } \sqrt{2\,gh_1 + v_p^2}.$$

Section II

STATICS

CHAPTER V

FORCE. MASS

In common usage the word "force" has been widely applied and greatly abused. In the preceding chapters we have tried to avoid using it, substituting for it such words as "influence," and "cause," in order that we might begin to use it at a time when we could consider its proper scientific meaning.

If two bar magnets, suspended so as to swing freely, are placed near each other, they will exhibit attraction or repulsion, depending upon the character of adjacent poles. Without stopping to theorize concerning the remote cause of this influence between the two magnets, we can accept the facts that there is an action in which both magnets participate — a mutual transaction, and that it results in motion or exhibits itself through the motion it produces.

This mutual transaction between the two given masses is called a force.

An electrified rod of glass, ebonite, or other non-conductive material (or even an insulated conductor when similarly electrified by rubbing or otherwise) will attract a stick or rod of metal suspended so as to swing freely. In fact, all the familiar phenomena of electrical attraction and repulsion are instances of an action between two masses, analogous to

the magnetic action above referred to. The remote cause of electrical attraction and repulsion may be very different from the cause of the analogous magnetic phenomena, but, nevertheless, in each case the ultimate effect is an action between the two bodies concerned in the transaction, which moves, or tends to move, both of them. We may have, therefore, a **magnetic force** or an **electrical force**.

If an elastic cord or a spiral spring be elongated, and a ball or other mass be fastened at each end, the cord or spring will contract again as soon as freed, and will even move the two inert masses. The rebound of a rubber ball is due to the same cause. In such cases the moving influence is an action of some sort between the particles of the rubber or of the material in the spring. But it is an action which produces motion, and hence is a force. It differs, however, from an action between two distinct bodies separated by an appreciable distance, as in the case of the bar magnets, because if the rubber band were stretched far enough to break it there would be no perceptible attraction between the two parts. It is some manifestation of cohesion between the individual particles or molecules of the rubber, and the sum total of these molecular actions is what we observe as the total force exerted by the rubber. It may be designated as a **molecular force**, or by means of any other adjective that will identify it through its origin or through any of its characteristics.

But the commonest exhibition of force is the **action of gravity**, with which we are all familiar in a general way. To our senses this action is evidenced through the phenomena of attraction between large masses, such as the general mass of the earth and objects on its surface, though the accepted theory of gravitation, which will be considered in a later paragraph, gives it a molecular origin. The theories of electrical and magnetic forces would put these also on a similar molecular basis. In all cases, however, whether the

size of the body or the quantity of matter under consideration be great or small, the force exhibits itself as an action between two masses.

From this point of view a **force** may be defined as a mutual action of attraction or repulsion between two masses.

The masses themselves may move under this influence, or the force may be employed to move some other mass, as the stirrups in which the magnets are suspended, or the two balls attached to the rubber cord, or an engine driven by the expansive force of steam.

Resistance. — If a person lifts a weight from the ground, the force exerted is a muscular contraction (similar to the contraction of a rubber band), which tends to pull the shoulder downward and the hand upward. If the weight is raised, we say that the resistance of gravity has been overcome. For the time being we cease, or forget, to think of gravity as an active agent; we regard the muscular force as the action and the force of gravity as a resistance, merely. But reverse the thought: if it were not for the resistance of the hand holding it, the weight would move to the earth, and so we think of the hand as resisting the action of gravity. When action is pitted against action, force against force, in this manner, it may suit our convenience to regard either as a resistance to the other.

When a force acts upon a body to bend it, or stretch it, or compress it, or otherwise deform it, there is occasioned in the body a resistance which had no existence until called into play by the action of the force itself. This is a case somewhat different from two entirely independent forces counteracting each other. For example, a body resting on a table is prevented from falling to the floor because the table resists, or stands against, the action of gravity. The attraction between the weight and the earth compresses the table and crowds its molecules together. The molecules

resist this displacement from their normal positions and exert among themselves an expansive force or repulsion, which causes the table to assume its original form and dimensions when the load is removed. Thus it appears that the elasticity of a body may be called upon to furnish a force that will resist an external force. Under such conditions the body is said to be subject to a *stress*. A stress may be a *pressure*, or a *tension*, or a *shearing stress*.

A body on a rough surface offers a frictional resistance to any force tending to move it along the surface. This, likewise, is always a passive resistance rather than a counteraction; it has no existence, or at any rate does not become evident, until called into play by the action of gravity or some other force.

These are typical illustrations of the phenomena and considerations to be dealt with in statics. A force, being a mutual action of attraction or repulsion between two masses, always *tends* to produce motion. If it is counteracted, or counterbalanced, in any manner whatsoever, it gives rise to **statical conditions.**

Action and Reaction. — The following law, from the time that it was first enunciated by Newton, viz. "To every action there is an equal and opposite reaction," has proved more or less misleading to beginners. Let us investigate some of the causes leading to a misunderstanding of this simple statement. Where there is but one force under consideration — a mutual or dual action between two masses — the application of the idea of "action and reaction" is quite simple. If two bodies, A and B, are connected by an elastic cord, the contraction of the cord pulls equally on A and B, in opposite directions. Or, if the force is not exerted by an elastic cord, but A and B attract each other, by gravity or otherwise, then this dual or mutual action may be regarded as a simultaneous operation of two separate reciprocal actions

— *A* upon *B*, and *B* upon *A*, and whichever we choose to mention first we may call the "action" and the other is its reciprocal action or its "reaction." To this extent the idea of "action and reaction" simply asserts the dual nature of a force.

It is in cases where one force is balanced by another, such as lifting a weight from the ground, that the complexities and difficulties arise. We have already stated that in lifting the weight the muscular contraction of the arm pulls down on the shoulder and upward on the hand, — equally and in opposite directions, like the elastic cord, — giving rise to two reciprocal operations, either of which may be taken as the action and the other as the reaction. This is still a consideration of a single force, and the idea is not yet confused. But, suppose we say that the downward tendency of the weight is balanced by the upward pull of the hand, as already stated in speaking of resistances: can we regard these two opposing influences as a case of "action and reaction"? It is here that the misunderstanding originates, — when the idea of "action and reaction" is extended to cover this class of cases which we have just termed resistances. It is true that these two opposing influences are equal and opposite, because they balance each other, but they cannot produce motion and do not constitute a force according to our definition; on the contrary, each is a part or member of two different forces counteracting each other. That is, in the case under consideration, we have one of the members of a second force — the upward muscular action on the hand — balancing the downward action of gravity on the weight. But this is only half the transaction; what becomes of the remaining member of each of these two forces? They must also balance each other, if the equilibrium between the two forces is to be complete. That is, if the upward muscular pull on the hand balances the downward pull of gravity on the weight, then the downward muscular pull on the shoul-

der must balance the upward pull of gravity on the earth. "The actions and reactions of the two forces equilibrate in pairs." Owing to the dual nature of each force, it requires a double transaction of resistances in order that the two forces may balance each other. In practice we do not generally take cognizance of more than one of these equilibrations at a time, but the other occurs, nevertheless.

These things are illustrated by diagram in Fig. 39, which shows:

First. — That each of the given forces — gravity and muscular contraction — involves an action and its reaction.

Second. — That these two forces balance each other completely, giving *two pairs of resistances* or equilibrations.

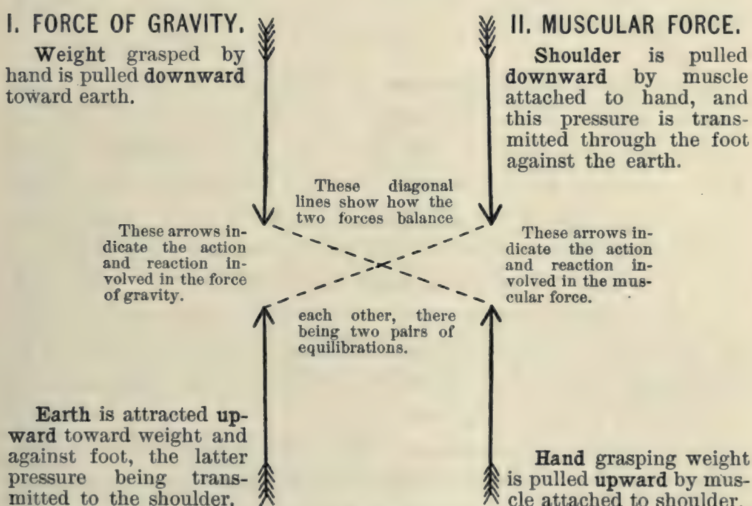


Fig. 39

Similarly, a weight resting on a table is sometimes cited as an instance of action and reaction. The weight pushes downward on the table and the table, it is asserted, "reacts" upon the weight. As a matter of fact the prime active

agency in the case is the force of gravity between the weight and the earth, so that if we start with the downward tendency of the weight and call it the "action," then its real reaction is the upward tendency of the earth. As each of these tendencies is overcome by the interposition of the table, through resistance of its molecules to compression, we have two pairs of equilibrations, — the top of the table *versus* the weight, and the bottom of the legs *versus* the earth. But to say that the table reacts against the weight, and the legs against the earth, is an objectionable use of the idea of action and reaction. It is confounding a *counter*-action with a *re*-action. The action and reaction, or force, between the weight and the earth is not destroyed or even suspended by the intervention of the table, but is merely restrained from producing motion.

Composition and Resolution of Forces. — A force is completely determined if its magnitude, direction, and point of application are known. A force can be represented in all these respects by a straight line. Component forces may be combined geometrically to determine their resultant, precisely after the manner of the "composition of velocities." If p, q, s, and t, Fig. 18 a, were forces instead of velocities, Fig. 18 b would be a polygon of forces instead of a polygon of velocities. This method is used most extensively in Graphical Statics, which deals with the determination of stresses in engineering structures.

For purposes of calculation the polygon can be cut up into triangles, as explained on p. 31, but a better method is to establish horizontal and vertical reference axes through the point of application of the forces, and then resolve each force into components parallel to these reference lines. All the horizontal and vertical components can then be combined by simple algebraic addition, giving two sides of a right triangle, from which to determine the resultant.

EXAMPLES

1. *Referring to Example 3, p. 32, assume that the components are forces (measured in pounds) instead of velocities. Compute their resultant by means of reference axes, in the manner just explained.*

2. *Do likewise for Example 4, p. 32, assuming that the numbers beside the components indicate kilograms.*

Law of Gravitation. — As already stated, the action of gravity is the most familiar example of force. The familiar facts of gravity are these:

The weight of a body depends upon the amount of material in it; the weight is directly proportional to the mass.

Forces of all kinds — electric, magnetic, muscular, etc. — are usually balanced by weights, and hence are measured in weight units, such as pounds, ounces, grams, kilograms, etc.

Weight acts vertically downward toward a point at or near the center of the earth.

The weight of a given body depends upon its distance from the center of the earth, being greater at the poles than at the equator, and greater near the sea level than at high altitudes.

The rotation of the earth diminishes the weight of bodies, with greatest effect at the equator and reducing to zero at the poles.

While these considerations may be clear in a general way, yet a proper appreciation of the law of gravitation requires considerable drill in the use of the so-called mathematical **law of inverse squares**.

Mathematical Expression of the Law of Gravitation. — The law of gravitation asserts that between every pair of particles in the universe there is a mutual attraction, which varies (1) directly as the product of the masses of the particles, and (2) inversely as the square of the distance between them.

(1) If A represents the force of attraction between two particles of matter, the masses of which are m and m', then

(disregarding distance for the present, and using the sign of variation to avoid the necessity of selecting units of force and mass), according to the first part of the law,

$$A \propto m \times m'.*$$

The particles m and m' may be alike, or they may be as different as we please; they may even be magnetized or electrified, and thus have a second attraction (or repulsion) entirely independent of gravity. Their gravitational attraction may be represented by diagram, thus:

FIG. 40

If m be replaced by two particles of the same kind, or by a single particle of twice the mass, the attraction will be doubled, thus:

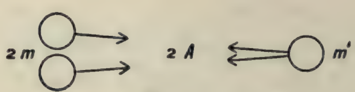


FIG. 41

If both m and m' be doubled, the attraction becomes four times as great (the distance, of course, being assumed to remain unchanged), thus:

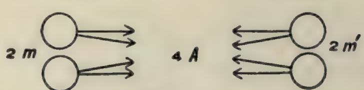


FIG. 42

If m be increased to $3\,m$ and m' to $5\,m'$, the attraction will become $15\,A$, etc.

(2) If the two original particles be not changed in mass, but are moved to different distances from each other, then as this distance d is changed

$$A \propto \frac{1}{d^2}.$$

* The symbol $\propto$ means "varies with."

If the attraction is A at distance d (Fig. 43),

FIG. 43

at distance $2d$ (Fig. 44) it will be $\frac{A}{4}$.

FIG. 44

At distance $3d$ it will be $\frac{A}{9}$; etc. If d be diminished to $\frac{1}{2}d$, it will be $4A$.

If the masses and distance both change simultaneously, what is the result? For instance, if m be doubled and m' tripled, and the distance doubled at the same time, the increase of masses will multiply the attraction six times, but the increase of distance will diminish the result to one fourth of what it would have been if the distance had remained unchanged. The result will be an attraction $\frac{6}{4}$ as great as the original A between m and m' at distance d. That is,

$$A \propto \frac{mm'}{d^2}.$$

The attraction between any two spherical masses of uniform density is the same in effect as if the entire material in each mass were condensed into its central particle. For instance, in computing the attraction between objects and the earth, we consider the distance to the center of the earth and not to the surface.

EXAMPLES

1. *A body, or aggregate of particles, has a mass of 8 units, and a second body has a mass of 15 units. At a distance of 12 units from each other, these bodies exhibit a gravitational attraction equal to 50 grams. Express the attraction between each of the following pairs of bodies at the distances specified:*

	Mass of Each of Given Bodies		Distance between Bodies
	I	II	
(*i*)	5 units	6 units	24 units
(*ii*)	5 units	6 units	48 units
(*iii*)	2 units	10 units	6 units
(*iv*)	30 units	3 units	2 units
(*v*)	40 units	7 units	8 units
(*vi*)	25 units	13 units	18 units

2. *The attraction between two masses,* m *and* 2 m, *at distance* d *from each other is equal to* A. *Another pair of magnitudes,* 2 m *and* 3 m, *are placed at a distance* 3 d *from each other; what is the attraction between them, expressed in terms of* A ?

3. *A body on the earth's surface weighs one pound.*

(*a*). *What will it weigh 4000 miles above the earth's surface ?*

(*b*) *What will it weigh 8000 miles above the earth's surface ?*

(*c*) *What will it weigh 6000 miles above the earth's surface ?*

It is assumed that the body is always weighed on a spring balance graduated at the earth's surface. Why ?

4. *The earth is said to be slowly cooling and contracting. How does this affect the weights of bodies on the earth's surface ? Why ? What would be the effect if the earth were expanding ? Why ?*

5. *Assuming that the mass of the moon is* $\frac{1}{81}$ *that of the earth, is there a point between them at which a body would have no weight ? If so, locate it.*

6. *A spherical body weighs 10 pounds at a given place on the earth's surface. Find the weight at the same place of a second sphere of twice the radius and three times the density of the first body.*

7. *A given body weighs one pound on the earth's surface.*

(*a*) *What will it weigh on another planet of the same density, but the mass of which is 27 times that of the earth ?*

(*b*) *What will it weigh on still another planet of the same density, but the mass of which is* $\frac{1}{64}$ *that of the earth ?*

8. *A spherical body weighs 100 pounds on the earth's surface. Find the weight of a second body of twice the radius and twice as dense, situated on the surface of a planet of three times the earth's radius, but only half as dense as the earth.*

Weight of a Body beneath the Earth's Surface. — It has been stated that a body carried above the earth's surface loses weight inversely as the square of its distance from the earth's center, but we should not infer from this that a body would gain weight if it could be carried in the opposite direction, toward the earth's center. On the contrary, it is obvious that if the earth were a sphere of uniform density, a particle at the center would be attracted equally in all directions, and furthermore, it can easily be demonstrated that a body in transit from the surface to the center would lose its weight uniformly with the distance. The weight of such a body would vary *directly as the distance* from the center, and *not inversely as the square of this distance.* While this demonstration involves practically nothing more than the law of gravitation, it gives rise to the application of the law of inverse squares to the following important proposition:

A body placed at any point within a hollow spherical shell of uniform density is at perfect equilibrium as regards the attraction of the shell.

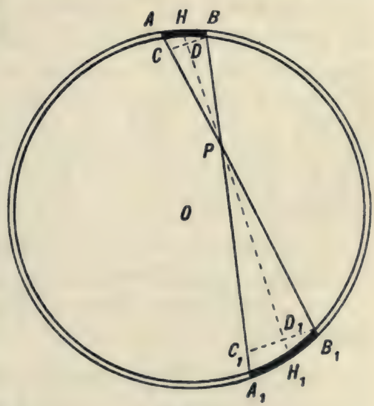


FIG. 45

In Fig. 45 let P be a particle at any point within the shell shown in section in the diagram. Let PB and PC represent elements of a right circular cone. If the surface of this cone were continued, it would cut out of the spherical shell a concave disk represented in section as AB. BP

and CP continued will form elements of a second and similar right circular cone, which would cut from the shell a second disk of section A_1B_1. (We shall speak of these sections in place of the areas of the disks which they represent.)

By geometry, the areas of the bases BC and B_1C_1 of these cones are to each other as $\overline{PD}^2$ is to $\overline{PD_1}^2$. As the vertex angle of the cone diminishes, the bases BC and B_1C_1 approach the areas AB and A_1B_1, respectively, as limits. Therefore, for any small area, as the limiting area AB, there is a corresponding area A_1B_1, so situated that

$$\frac{\text{area } AB}{\text{area } A_1B_1} = \frac{\overline{PH}^2}{\overline{PH_1}^2}.$$

Under these circumstances it is obvious that the attraction between the particle at P and the mass of AB is just equal and opposite to the attraction between the particle at P and the mass of A_1B_1. For instance, if the mass included within the area A_1B_1 is four times as great as the mass of AB, it can be so only because the *square* of the distance PH_1 is also four times the *square* of PH; the predominance of mass A_1B_1 is exactly neutralized by its greater distance (squared).

If it were not that this relation of dimensions of the spherical shell always coincides in effect with the law of inverse squares as applied to attractions, a particle would not be at equilibrium at any point within the shell, except at the very center; no other law of attraction would give the general condition of equilibrium for all points within the spherical shell.

An electrified pith ball remains at rest at any point within an electrified spherical shell, showing that the law of inverse squares is also applicable to attractions and repulsions between electrified bodies.

Now, if a body is carried towards the center of the earth, at any point *en route*, as at A in Fig. 46, the entire spherical

shell of thickness AB can be regarded as exerting no attraction whatever upon the body. The resultant attraction upon the body is the same as if the earth were a planet of radius OA. If $OA = \frac{OB}{2}$ and the density of the earth were uniform at all depths, then the mass of the shaded portion would be $\frac{1}{8}$ of the total mass of the earth. Hence, while the body at A would be only 2000 miles (or $\frac{1}{2}$ as far as B) from the earth's center, on the other hand it would be attracted by only $\frac{1}{8}$ of the mass, and the apparent weight would be $2^2 \times \frac{1}{8}$, or $\frac{1}{2}$ of its weight at the surface.

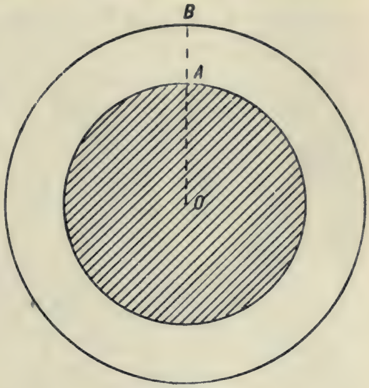


FIG. 46

That is, the portion of the earth which acts as an attracting mass varies directly as $\overline{OA}^3$, and the effect of distance inversely as $\overline{OA}^2$; hence the weight varies as $\frac{\overline{OA}^3}{\overline{OA}^2}$, or directly as OA.

In Example 7, page 76, in which a body was weighed on different planets, the weight of the body varied directly as the radius of the planet.

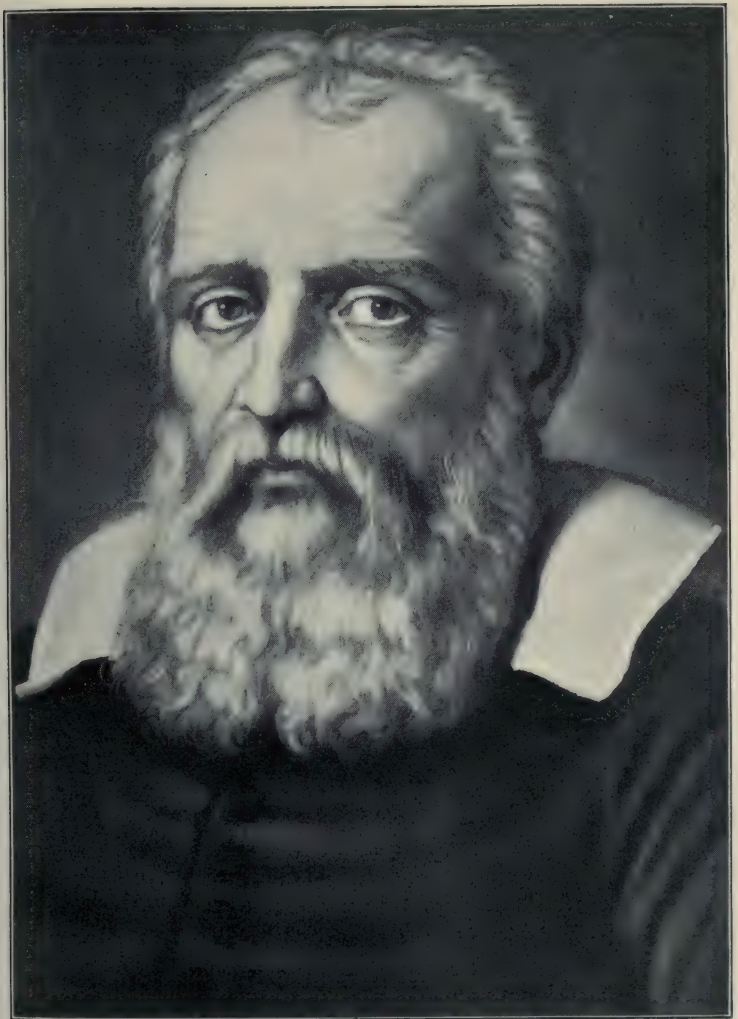
This demonstration assumes that the earth is of uniform density at all depths, but the fact is that the density increases toward the center. Observations indicate that bodies carried downward exhibit an actual increase of weight instead of a decrease, down to a certain depth, beyond which the weight begins to diminish. It appears, therefore, that a body really has its maximum weight, not at the surface, but a short distance below.

Mass. — As stated in the Introduction, page 8, we have not considered a precise definition of mass, though we have used the related ideas of weight, density, and specific gravity.

Our use of this word, therefore, has been more or less tentative, and even now we cannot enter into a full discussion of its meaning because it has much to do with kinetics and that branch of mechanics we are to study later. Our assumption has been that every body has in it a definite amount of matter, and one way to judge this amount is by the weight of the body. If we could be suddenly transported to the surface of the moon with the given body, its apparent heaviness would be greatly diminished, though its mass has remained unchanged. In fact, as we learned from the law of gravitation, the mass of a piece of lead even would have no weight at all if it were not for the existence of some other attracting mass, such as the earth. But we cannot conceive of the lead losing its mass; the very existence of the body is made known to our senses through its mass, or the amount of material existing in it. For this reason mass is said to be one of the *essential* properties of matter, — essential to our conception of its existence.

Masses are designated as pounds, ounces, grams, kilograms, etc., — the same as weights and forces. This identity of names as applied to different things arises, of course, from relations between those things, but it is confusing to a beginner who has not yet learned those relations. According to the law of gravitation the weight of a body at any given place (that is, its attraction toward the earth) is proportional to its mass. Expressed mathematically, *the weight* $\propto$ *the mass*. If we choose, we may take a lump of any kind and call it a unit mass, and then we may take as the unit weight the weight of this unit mass, or of any fraction or multiple of it, at any standard place as regards latitude and altitude. A lump of platinum * preserved at the Archives of Paris as a standard mass of one kilogram also *weighs* one

* This is supposed to contain the same mass as a liter of water at its temperature of maximum density, 3°.9 C., but it has been found that a liter of pure water at 3°.9 C. really weighs 1.000013 Kg.



GALILEO

kilogram *at Paris;* it weighs more than a kilogram at the north pole, and less than a kilogram at the equator.

The standard pound mass is a piece of platinum preserved in the office of the Exchequer at London; the weight of this mass *at London* is the standard pound weight. Technically this is called the imperial pound, and it is very nearly equal to the older avoirdupois pound from which it was intended to differ slightly for certain purposes of correction.

Density. — The density of a given substance is the **mass of a unit volume** of that substance — expressed in grams per cubic centimeters, pounds per cubic foot, etc. Notice that it is the mass of a unit volume, and not the weight. The weight of the body may be different at different places; it may even be zero, as at the center of the earth, in which case if the density were regarded as the *weight* of a unit volume, we would have to imagine the body to exist without density.

Heavy and Light Bodies fall at the Same Rate. — In discussing the acceleration of gravity we assumed that all bodies fall at the same rate. This is true if we disregard the frictional resistance due to the presence of the air. Obviously, a light feathery body or a piece of paper has to displace and drag itself along through a quantity of air that is very large in proportion to the weight of the body. In a vacuum a feather falls as fast as a piece of metal.

A little thought will show that it would not be consistent with our common experience for a heavy body to fall faster than one half as heavy. If two bricks are dropped, one from each hand, they will reach the ground in the same time; if the two had been glued together, there would be nothing to make the double mass fall any faster than the single bricks. In kinetics this would be explained by saying that while the double mass has twice the attraction downward, it also has twice the mass to be moved by that attraction.

CHAPTER VI

WORK. POWER. ENERGY

We know that to lift a weight or drag a vehicle, or to overcome a resistance of any kind, requires the application of a force. If we consider this applied force in relation to the distance through which the body moves under its influence, the product of these two values — the force and the distance — introduces one of the most important conceptions in mechanics. In lifting a pound weight one foot, it is said that one **foot-pound of work** is done. To lift a 5-pound weight from the floor to the top of a table 3 feet high requires the performance of 15 foot-pounds of work. If a vehicle requires a pull of 130 pounds to drag it along a horizontal surface at a uniform rate, the work done for every mile that it is moved along such a surface is 130×5280 foot-pounds.

Of course, there may be many other units of work, as a mile-pound or a mile-ton; a kilogram-meter or a gram-centimeter, according to convenience for large and small measurements, and to meet other needs. For most purposes, however, it is seldom that any unit of work is employed other than the foot-pound for the English system, and the kilogram-meter for the metric system.

The work is said to be done *by* the applied force and *against* the resistance overcome.

The time consumed is not a factor in determining the amount of work done in overcoming a resistance through a certain distance. The work done in carrying a thousand bricks to the top of a given building is the same whether it

is accomplished in a day or a week. Work would be done faster in one case than in the other, but the total number of foot-pounds is the same.

From the standpoint of mechanics, no work is done when a weight is held at rest in the hand, or when it is moved horizontally with a uniform velocity. In the first instance, it is obvious that the body is moved through no distance, so that the force multiplied by the distance is *zero*. In the second case, the body is moved, but it is neither raised nor lowered. The amount of work done is determined by the distance through which the resistance yields to the applied force. With the weight held in the hand the muscular action and the action of gravity are both vertical, and as there is no vertical motion there is no work done. The work done in carrying a weight up a flight of stairs is the same as if the body were lifted vertically from the lower floor to the next. Work done against gravity varies directly with difference of level.

To make this point entirely clear, it seems almost necessary to anticipate a principle of kinetics. It was stated on page 59 that, if it were not for the action of gravity, a free and unobstructed body hurled in any direction would continue to move in a straight line with uniform velocity; that is, no force would be required to keep it moving, and hence no work would be done as it moved mile after mile through space. Even with gravity acting, if it were not for friction, it would not require work to keep a vehicle moving at a uniform rate on a horizontal plane, once it were started. On a *perfectly smooth* horizontal plane (if such could be realized) the slightest force would start any mass, howsoever large; as long as this force continued to act, the body would keep on moving faster and faster; when the desired speed is attained, the force could be withdrawn, or cease to act, and the body would continue to move on indefinitely with this velocity. In other words, when a mass is moved on a horizontal surface it is the resistance of friction that must be overcome and against which work is done; the attraction between the mass and the

earth is not directly overcome, and hence no work is done against it. Indirectly the weight counts in this way, that the friction encountered in moving a body is directly proportional to its weight, and for that reason more work would be done in moving a heavy body than in moving a lighter one through equal distances on the same horizontal plane.

EXAMPLES

1. *A person weighing 150 pounds walks up a flight of stairs between two floors 14 feet apart. How much work is done? What if a vertical ladder had been used? What if he had climbed up on a rope?*

2. *A horse drags a plow half a mile against a resistance of 200 pounds, as indicated on a dynamometer. How much work is done?*

3. *A person walking against the wind has to overcome a resistance of 4 pounds per square foot. If the surface meeting this resistance is 5 square feet, how much work does he do for every mile walked?*

4. *A gallon contains 231 cubic inches. A cubic foot of water weighs 62.5 pounds. What work will be required to pump 1000 gallons of water through a vertical height of 50 feet?*

5. *How many foot-pounds of work will be done in raising five gallons of alcohol from the floor to the top of a table 33 inches high? (Specific gravity of alcohol = 0.8.)*

6. *A person weighing 155 pounds rides up a hill on a bicycle weighing 27 pounds. If the hill rises 10° from the horizontal, what work is done for every 100 feet ridden, neglecting friction?*

7. *In a steam engine of 9.5-inch diameter of piston and 12-inch stroke, how much work is done during each complete stroke if the effective steam pressure in the cylinder averages 30 pounds per square inch? If the engine is running 200 revolutions per minute, how many foot-pounds of work are done in a second?*

8. *A kilogram-meter is equivalent to how many foot-pounds?*

Power. — The horse power of an engine is determined by the *rate* at which it can do work. The idea of power simply

modifies the idea of work by introducing a time element. (Some writers use the word "activity" instead of power.) To raise a ton of granite to the top of a building 75 feet high requires 150,000 foot-pounds of work, without regard to the time consumed. But the engine that can accomplish this amount of work in one minute has twice the power of an engine that would require two minutes for the same work.

A natural or logical unit of power would be the ability to do one foot-pound of work per second, or some similar value employing any convenient units of force, distance, and time. It is the practice, however, to use an entirely arbitrary unit, — **the horse power.** An engine or other agency works at the rate of one H.P. if it performs 550 foot-pounds per second, or 33,000 foot-pounds per minute.

The horse-power of an engine under a given pressure of steam could be readily computed from its dimensions and speed if the full steam pressure acted throughout the entire length of stroke, or even if the average effective pressure were known. For example, assume the following conditions: diameter of piston 9.5 inches; length of stroke, 12 inches; number of revolutions, 200 per minute; boiler pressure, 75 pounds per square inch. If the full steam pressure acted throughout the entire stroke, the work done during each double stroke, or revolution, would be

$$2 \times 75 \times \frac{\pi \times \overline{9.5}^2}{4} \times \frac{12}{12} \text{ foot-pounds.*}$$

The work done per minute would be

$$2 \times 75 \times \frac{\pi \times \overline{9.5}^2}{4} \times \frac{12}{12} \times 200 \text{ foot-pounds,}$$

and the horse-power supplied to the piston from the energy of the steam would be

$$2 \times 75 \times \frac{\pi \times \overline{9.5}^2}{4} \times \frac{12}{12} \times 200 \times \frac{1}{33000}, \text{ or } 64.44 \text{ H.P.}$$

As a matter of fact, the mean effective pressure on the piston is considerably less than the pressure of the steam as it enters the

* Assume $\pi = 3.1416$.

cylinder. The inlet valve of an engine is adjusted in such a manner that the steam supply is shut off when the piston has completed only a fraction of its stroke — a third, or a fourth, or at whatever point may be necessary for the best economy under the given conditions of steam pressure and load. After the steam is cut off, the remaining part of the stroke must be completed by the expansion of the steam previously supplied, and, of course, the pressure of this steam diminishes as its volume (the space it occupies in the cylinder) increases.

It is the function of the governor to control the supply of steam as needed to maintain a constant number of revolutions per minute. There are two types of such governors. If the valve is fixed so that it always cuts off the supply at the same fractional part of a stroke, then the governor regulates the steam supply by varying the size of an opening through which the steam is made to pass *en route* to the cylinder. The common Watt governor is of this type. In an automatic cut-off engine the governor controls the valves themselves, varying the point of cut off.

The valves of an engine are also adjusted to control the outlet of steam, or the exhaust. To accomplish easy running, enough steam should be confined in the idle end of the cylinder at each stroke to furnish a cushion and thus prevent a sudden stopping and consequent jarring at the end of the stroke. To compress this steam, of course, neutralizes some of the pressure in the active end of the cylinder, so that the mean effective pressure is still further reduced from this cause.

In the example selected for illustration, on the preceding page, it was found that the steam would convey to the engine 64.4 H.P., if the total pressure of 75 pounds per square inch were to act throughout the entire stroke of the piston. Now, on page 56, it was stated that this same engine, under exactly the same conditions, developed 20.26 H.P., as determined from an indicator card. This shows that the mean effective pressure on the piston was less than the boiler pressure in the ratio of $\frac{20.26}{64.44}$. If P_m is the mean effective pressure, then

$$P_m \times \frac{\pi \times \overline{9.5}^2}{4} \times \frac{12}{12} \times 2 \times 200 \times \frac{1}{33000} = 20.26,$$

or $P_m = 23.59$ pounds per square inch, which will be found to be $\frac{20.26}{64.44}$ of 75.

Some of the energy transferred to the piston from the steam is wasted by friction in the engine itself. The rate at which energy is supplied to the piston is called the indicated horse power, as contrasted with the actual horse power available from the engine.

The ratio of the actual horse power to the indicated horse power, expressed as a percentage, is the efficiency of the engine or the ratio of output to input.

Electrical power is measured in **watts**, this unit being the rate at which energy is conveyed by a current of one ampere intensity under a potential of one volt. One H.P. in mechanical measure is equivalent to 746 watts.

EXAMPLES

1. *How many foot-pounds of work can be done in 8 hours by a 50-H.P. engine ?*

2. *How many tons (2000 pounds) of coal can be raised per hour from the ground to a bin 60 feet above by a 10-H.P. engine ?*

3. *What H.P. will be required to pump water to a height of 120 feet at the rate of 1000 gallons per minute ?*

Assume 231 cubic inches to a gallon and 62.5 pounds as the weight of a cubic foot of water.

4. *If a horse can perform 550 foot-pounds of work per second, at what rate in miles per hour can he drag a plow against a resistance of 225 pounds ?*

5. *A person carries a sign measuring 3 feet by 4 feet against the wind, the resistance being 6 pounds per square foot. If he walks 2 miles per hour, at what H.P. is he doing work ?*

6. *A person weighing 150 pounds runs up a flight of stairs in 5 seconds. If the stairs are 27 feet long and rise at an angle of 34°, the person is doing work at the rate of what H.P. ?*

7. *How many gallons of water will a 40-H.P. engine pump in an hour from a mine 500 feet deep ?*

8. *A house on rollers is moved by means of pulleys and a windlass. If the resistance to rolling is 20 tons, at what rate can it be moved by a single horse working at the rate of one H.P.?*

9. *What must be the H.P. of an engine if it is to be used for running a 110-volt dynamo that generates a current of 50 amperes?*

10. *What is the H.P. of an electric motor driven by a current of 75 amperes under potential of 125 volts?*

11. *Compute the H.P developed by an engine under the following conditions: diameter of piston, 4 inches; length of stroke, 5 inches; mean effective pressure, 42 pounds per square inch; number of revolutions, 275 per minute.*

12. *An engine has the following dimensions: diameter of piston, 12 inches; length of stroke, 18 inches. At 150 revolutions per minute the indicated H.P. is 35.2. Find the mean effective pressure.*

Energy. — It has been stated that work is regarded as being done *by* the applied force and *against* the resistance. When a person lifts a weight, the applied force is clearly the muscular action and work is done against the resistance of gravity. But suppose that in this elevated position the weight is attached to a clock or other piece of mechanism; by the action of gravity the weight descends and does work against the resistance of the machine driven by it. If it is a 10-pound weight and was lifted 2 feet above the floor, the work done in raising it was 20 foot-pounds; this is exactly equal to the amount of work that can be done on the machinery by the weight in returning to the floor. The work done in lifting the weight was not wasted; it was simply vested in the elevated mass as available energy, ready to be given back in full by performing 20 foot-pounds of work. A body that is in any way endowed with the **ability to do work** is said to possess **energy**. If it can do 100 foot-pounds of work, it possesses 100 foot-pounds of energy.

Potential Energy. — Hydraulic elevators are operated by storing water in an elevated tank or otherwise subjecting it to pressure. Many machines are driven by compressed air, the work done in compressing the air being given back as the molecules return to their normal distances from each other. The energy stored in a clock spring by winding is of a sort much the same as the energy of compressed air; in all such cases advantage is taken of the elasticity of the material, which offers a resistance to any force tending to deform it and thus stores up any work done upon it. These are all examples of potential energy or, as it is sometimes called, **energy of position**.

Two bodies that attract each other, such as a weight and the earth, are endowed with this energy only by the act of separating them. The energy vested in the bent spring or the compressed air, or in any other elastic body, is due to the displacement of the particles from their normal positions. It is the tendency of the disturbed bodies, or the disturbed particles of the distorted body, to return to their normal positions that gives them the ability to do work.

The adjective "potential" means "possible," and is used to signify that the exhibition of this energy is contingent upon the condition that the displaced body or particles be allowed to return to their normal position or positions. For instance, if the elevated weight is placed on a level shelf or a table-top, it can do no work in that position, but must be allowed to fall to the ground if its energy is to be exhibited. The driving weight of a clock is held by a catch and performs its work only as it is released by the escapement. The energy of a bent spring and of compressed air is potential because of the same contingency.

The potential energy of a body, as ordinarily understood, is not an absolute quantity. The potential energy of an elevated weight, for example, is measured relatively to the floor, or the ground, or some other plane taken as a standard.

Kinetic Energy. — As contrasted with potential energy or energy of position, a body may have kinetic energy **due to its velocity**. For illustration, a jet of water can be made to drive a water wheel and thus perform work; the work done by a windmill in pumping water is readily traced to the energy of the air current; it is the kinetic energy of the carpenter's hammer that does the work of driving a nail.

Energy of motion differs from energy of position, also, in this respect, that the former exists free from all contingency. For that reason kinetic energy is sometimes called **actual energy,** in distinction from potential (or possible) energy.

For a given velocity, the kinetic energy of a body is the same no matter what the direction of motion. It could be *measured* experimentally by attaching the body in such a way as to compel it to overcome a known resistance or by letting it strike some properly arranged obstacle, and then observing how far the resistance yields before the body comes to rest. The simplest way to *compute* the kinetic energy of a body from its mass and velocity is to find out how far it would rise *if it were moving vertically upward.*

For example, suppose a given body to have a velocity of 96.6 feet per second, and we wish to find its kinetic energy. Now we know from the principles of acceleration that this body, if it were moving vertically upward and started from the ground with this velocity, would rise to a height of 144.9 feet. In other words, by virtue of its velocity, it would be able to lift itself to this height against the action of gravity. Its weight multiplied by this height is, therefore, the amount of work it can do, or its energy.

As a further example, if a bicyclist wished to know his kinetic energy at a certain speed, he could start up a hill of known pitch with this velocity and, removing his feet from the pedals, note how far he is carried up the hill. His weight multiplied by the *vertical* height to which he ascends will represent his kinetic energy at the foot of the hill.

In Formula 8, page 53, it was proved that the distance a body would travel while gaining (or losing) a given velocity varies as the square of this velocity. Hence the kinetic energy of a body depends upon the square of its velocity. One body traveling 5 times as fast as another body of the same weight would have 25 times as much kinetic energy.

In heat, sound, light, and electricity, we find examples of **molecular energy**, as distinguished from **mechanical energy**, which heretofore we have drawn upon exclusively for purposes of illustration. "The heat possessed by a body is explained as being the energy possessed by it in virtue of the motion of its particles. Just as a swarm of insects may remain nearly at the same spot while each individual insect is energetically bustling about, so a warm body is conceived as an aggregation of particles which are in active motion while the mass as a whole has no motion." * Such a body has invisible molecular kinetic energy. Moving *en masse*, with visible motion, its energy would be called mechanical. Obviously, it may have both at the same time. Or, if raised to an elevated position, it would also have mechanical, potential energy — its molecular energy, of course, remaining unchanged.

The **chemical energy** of gunpowder is a familiar example of *potential* molecular energy, though perhaps it would better be called atomic rather than molecular. The different constituents of gunpowder—sulphur, saltpeter, and charcoal — have a chemical affinity for each other, by virtue of which they tend to come together and form new compounds, just as a weight tends to fall to the earth. In the gunpowder the constituents, while intimately mixed together, are still separated from each other chemically, and hence they have potential energy. Once the impulse is given, like releasing the elevated weight from the shelf, these chemical constituents rush together, atom clashing against atom, forming new

* Daniell's "Text-Book of the Principles of Physics," p. 48.

gaseous compounds, and generating an enormous amount of heat. Some of the original chemical energy (molecular potential) is thus converted into heat (molecular kinetic).

Any form of **molecular energy is measurable in mechanical units.**

The watt, as already stated, is equivalent to $\frac{1}{746}$ H.P. This is really a conversion of electrical into mechanical *power*, and *not* a conversion of *energy*. The mechanical equivalent of an electrical unit of energy could easily be derived, but is not needed in dealing with *currents* of electricity, because a current always involves a time element, which we also found to be involved in power but not in work.

The unit of heat is the amount of heat necessary to raise a unit mass of water one degree in temperature. Taking a pound as the unit of mass and using the Fahrenheit thermometric scale, the amount of heat necessary to raise a pound of water 1° F. is equivalent to 778.5 foot-pounds of work. In the metric system the heat necessary to raise a gram of water 1° C. is called a "lesser calorie"; to raise a kilogram of water 1° C. a "greater Calorie."

Transference and Transformations of Energy. — Any one of the different kinds of energy can be transformed, directly or indirectly, into any other kind. Mechanical energy is readily changed from the kinetic form to the potential, and *vice versa*. Mechanical energy of either form can be converted into heat, sound, light, or electricity. And any one of these various molecular forms can be transformed into any other, or into mechanical energy. Chemical energy is likewise convertible, especially into heat and electricity.

Energy can also be transferred from one body to another and thus transmitted from place to place. Whenever work is done there is either a transformation or a transference of energy, or both. In fact, at this point our **definition of work** might well be revised in accordance with this larger view.

But with all these changes of energy from body to body and from one form to another, there is no actual gain or loss of energy. It is always a case of adding and subtracting the same quantity. The work done *by* one body exerting a force on a second body is just equal to the work done *upon* this resisting body, or in other words the first body yields up to the second body an amount of energy equal to the number of foot-pounds of work done. So far as we know the total energy of the universe is an unchanging quantity — a definite number of foot-pounds. This hypothesis is known as the **conservation of energy**, and is accepted as one of the most fundamental principles of physics.

A simple illustration is in the conversion of mechanical energy from the kinetic form to the potential, and *vice versa*. Suppose that a 10-pound weight is projected vertically upward with a velocity of 96.6 feet per second. At the instant of leaving the ground it has 10×144.9, or 1449 foot-pounds of *kinetic* energy, because by virtue of its velocity of projection it is capable of rising to a height of 144.9 feet against a gravitational resistance of 10 pounds. But when it has reached the highest point, ready to start downward — what then? In this position it has 1449 foot-pounds of *potential* energy. And if at that instant the weight could be caught on a hook or a shelf, it could then be attached to a clock or other mechanism and its potential energy used for any purpose desired, in which event the body would return to the ground leisurely and perhaps with a velocity hardly perceptible. But if it is not thus restrained, but is allowed to fall freely from this height of 144.9 feet, it will fall in the usual manner at an accelerated rate. On the downward trip it will gain velocity as fast as it lost it on the upward trip, and when it reaches the ground it will have the same kinetic energy that it started with. The instant it strikes the ground, however, this mechanical energy of motion is converted into sound, heat, and perhaps light and electricity. It follows, therefore, that when a body is projected upward its kinetic energy is *gradually* converted into potential energy, changing back to the kinetic form as it descends. At the

summit it has only potential energy. For every foot that it rises, if its weight is 10 pounds, it gains 10 foot-pounds of potential energy and loses an equal amount of kinetic energy. Halfway up half its energy is potential and half kinetic. The total energy, however, remains the same.

The generation of an electric current affords another good illustration of transformation. The current itself is, of course, an example of transmission or transference of energy. If the current is generated by a battery — a voltaic current, it is a case of direct conversion of chemical into electrical energy. If it is furnished by a dynamo — a magnetic-electric current, its energy can be traced back step by step, first to the mechanical energy of the machinery, and thence to the energy of the steam generated by the heat of the furnace. The heat itself originated in chemical energy of the coal and oxygen uniting with each other to form carbonic acid and other gases. We might go a step further and say that the energy of the coal and oxygen came from the rays of the sun which in some past age decomposed a lot of carbonic acid gas, giving the carbon to make the wood of a tree and liberating the oxygen to form part of the atmosphere.

This dynamo might have been driven by a water wheel, power being furnished by a jet of water. And perhaps the water was first stored for some length of time in a reservoir. Looking still further we find that it was raised to the reservoir from the ocean at the expense of energy of the sun.

There is hardly an instance in which the power used by man could not be traced to the sun. The energy radiated from the sun itself may be due, all or in part, to chemical action now going on there or to radiation from a molten mass, or to still other causes equally probable and based upon even more recent theories.

EXAMPLES

1. *What is the potential energy of a body weighing 100 kilograms at an elevation of 22 meters?*

2. *A body weighing 5 pounds is dropped from a height of 1000 feet. What kind of energy and how much did it have when it started? At the end of one second what change will have taken place? At the*

end of two seconds? At the end of three seconds? Just as it reaches the ground? After it has struck the ground?

3. *The same body is projected vertically upward with a velocity of 322 feet per second. How much energy has it and of what kind? What change will have taken place at the end of one second? At the end of two seconds? At the end of three seconds? At the end of ten seconds?*

4. *What transformations of energy take place during the oscillations of a pendulum?*

5. *A bicyclist weighing 155 pounds, mounted on a 27-pound wheel, rides at a velocity of 20 miles an hour on a horizontal plane.*

(*a*) *What is the combined kinetic energy of the man and wheel?*

(*b*) *If he comes to a slope rising at an angle of 12° and removes his feet from the pedals, how far will he progress up the hill?*

6. *The same person riding at the same rate comes to a hill of unknown pitch, and finds that his velocity is sufficient to carry him 75 feet measured along the slope. Find the pitch of the hill from the horizontal.*

7. *In the Yosemite Water Fall the total drop is about 3000 feet.*

(*a*) *What transformations of energy occur?*

(*b*) *When the water strikes at the foot of the Falls its mechanical kinetic energy is converted mainly into heat. If there were no loss from friction, etc., during the drop, and if all the energy were converted into heat at the instant of striking, how much would the temperature of the water be raised?*

8. *A body has a velocity of 100 feet per second and weighs 21 pounds. What kinetic energy has it?*

9. *A body weighing 50 pounds and having a velocity of 40 feet per second moves along a horizontal plane against a frictional resistance of 2 pounds.*

(*a*) *How far will it travel before coming to rest?*

(*b*) *How long before it will come to rest?*

HINT. — First compute the energy of the body by finding out how far it would rise against its own weight *if it were moving vertically upward.*

10. *A freight car weighing 30 tons is moving on a horizontal track at the rate of 40 miles per hour. If the total resistance occasioned by the breaks and by friction on the tracks is 2 tons, how far will the car move before it is brought to rest?*

11. *What transformations of energy take place when a firecracker explodes? When a steam whistle blows? When a gong is sounded?*

12. *An electric current is used to drive a motor attached to a lathe. What transformation of energy takes place? What if the current had been used for an incandescent lamp? For electroplating? For ringing an electric bell? For electric welding?*

Graphical Representation of Work. — Using two rectangular axes in the manner explained on pages 55 and 56, work can be represented as an area — a rectangle, of which one side or dimension stands for either the applied force or the resistance, and the other dimension stands for the distance through which the resistance is displaced. For example, to represent the work done in raising a weight of 600 pounds to a height of 50 feet, let us assume a vertical scale of 1 inch = 200 pounds and a horizontal scale of 1 inch = 10 feet. From the origin O, Fig. 47, measure off on the Y-axis the length $OA = 3$ inches (to represent the weight, 600 pounds), and on the X-axis measure off the distance $OB = 5$ inches (to represent the displacement of 50 feet).* Since OA represents a force and OB a displacement caused by this force, it follows that the work done is represented in the diagram by the area of the rectangle $OBCA$, or what is the same thing $OA \times OB$.

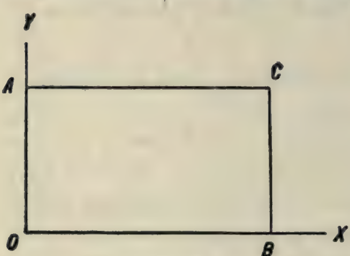


FIG. 47

Since the weight remains constant throughout the displacement, a perpendicular erected at B or any point between O and B, and equal to OA, will represent the magnitude of the force for that particular point in the displacement. The *locus* of the

* The printed diagram is reduced to one quarter of this scale.

extremities of such perpendiculars will be a line through point A parallel to OB.

If the force had been variable, the *locus* AC would be an irregular line as yy_1 in Fig. 36, page 58. The work done, however, would still be the area included between this line and the two axes, or Oxy_1y in Fig. 36. In fact, this general method of interpreting an area is the same as was explained on pages 57 and 58 in connection with accelerated motion, except, of course, that the coördinates in that case were made to represent entirely different quantities from those we are now dealing with in connection with work.

Between these two extreme cases of a uniform force and an irregularly changing one, there are circumstances under which the force changes uniformly, and for which the diagram yields a **triangular area** to represent the work done. This is always the case when a force is applied so as to gradually elongate a wire or rod, or a spiral spring, and in all other cases **where a stress is produced in a body** — as in bending a beam, or twisting a shaft, or compressing a block. Imagine a small load, say a pound weight, applied to a spiral spring; the spring elongates a certain amount and stops, showing that its ability to resist the elongating force gradually increases as it is stretched. If the load be doubled by adding a second weight, the spring will stretch still farther, — until the internal resistance again becomes equal to this external load. Now if we measure the length of the wire under this load we will find that the elongation for the two pounds has been just twice as great as for one. If we add a third pound, the elongation becomes three times as great, the internal resistance, of course, increasing in the same ratio in order to balance the weight. In other words, the internal resistance of the spring is directly proportional to the elongation. The same idea applies to all bodies, whatever the nature of the stress. This is called **Hooke's law**, which asserts that within certain limits the internal resistance of a distorted body is directly proportional to the amount of distortion or deformation. When the internal resistance becomes equal to the external force, the distortion ceases; and hence for bodies at equilibrium under stress — the usual

case — the law might be expressed in the converse form, viz., that the distortion is directly proportional to the distorting force.

In graphical form the work done in elongating the spring by adding successive weights of one pound each would be represented as in Fig. 48 *a*. Let OA' represent the force of one pound and OB' the elongation produced by this force; the work done thereby will be the area $OB'C'A'$. Then a second weight was added, making two pounds in all, as $B'A''$, and producing an additional elongation $B'B''$, or a total of OB'' equal to 2 OB'. As we go on stretching the spring farther and farther, the area representing the work increases step by step in the manner pictured in the diagram. The work done in stretching the spring through the distance represented by $B'''B''''$ is four times as great as in stretching it through the same distance OB' at the first stage of the elongation.

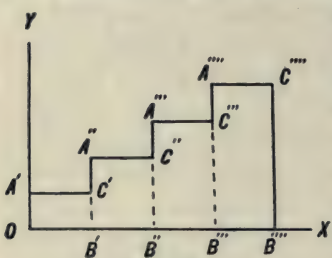


FIG. 48 *a*

If the load had been applied to the spring gradually, increasing continuously instead of a pound at a time, the diagram would have been as in Fig. 48 *b*. If the load increases uniformly from zero to BC, while the spring is elongated an amount represented by OB, the work done will be represented by the area of the triangle OBC, which is only half as great as if the maximum load BC acted throughout the entire distance, as in Fig. 47. Or, the average load is only a half of BC.

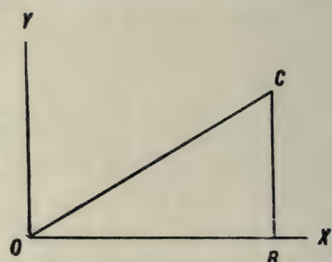


FIG. 48 *b*

What is true in this respect for the elongation of a spring is true for a body distorted in any manner.

Boyle's law (or Mariotte's law, as it is sometimes called) asserts that the volume of a gas varies inversely as its pressure — or, in other words, that the volume times the pressure is constant —

provided the temperature remains unchanged. Suppose that a cylinder, Fig. 49, is fitted with a piston, by means of which a quantity of air is confined in the cylinder. When the pressure on the gas is 90 pounds, suppose that the volume is 6 cubic inches. Then, as the pressure is changed, the volume can be determined from the relation $vp = 540$, or $v = 540/p$. The upper part of the diagram shows two reference axes and a curve, illustrating the law. Horizontally, let 1 inch equal a volume of 12 cubic inches and vertically let 1 inch equal 90 pounds pressure. From the equation $v = 540/p$, the following table of values is obtained:

When $p = 0$	$v = \infty$
When $p = 18$	$v = 30$
When $p = 22.5$	$v = 24$
When $p = 30$	$v = 18$
When $p = 45$	$v = 12$
When $p = 90$	$v = 6$
When $p = 180$	$v = 3$
When $p = 270$	$v = 2$
When $p = \infty$	$v = 0$

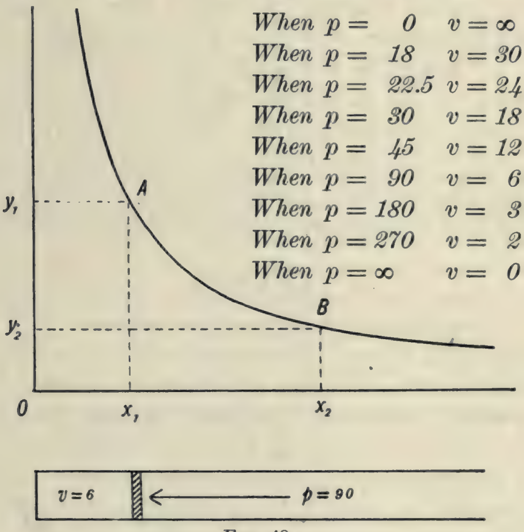


FIG. 49

From this table (interpolating values if necessary) we may plot a curve, as shown in the figure, and from this curve we can make several important deductions.

First, if we take any two points A and B on the curve and through them draw lines parallel to the axes, than the rectangles Oy_1Ax_1 and Oy_2Bx_2 are equal in area. Oy_1 and Ox_1 are the pressure and volume respectively of the gas for point A, and Oy_2 and Ox_2 are the pressure and volume for point B; and the product of pressure and volume is the same for all points.

Second, the area ABx_2x_1 represents the work done by the expansion of the gas from volume v_1 to v_2, or conversely, the work necessary to compress the gas from v_2 to v_1. The average pressure doing this work is represented by the average of all the vertical lines that could be drawn between x_1A and x_2B, and this average pressure multiplied by x_1x_2, the distance the piston is moved, is the work done.

Notice how the outline y_1ABx_2 resembles the corresponding portion of the indicator diagram shown in Fig. 50, page 101. Notice, also, an important difference between the graphical results of Hooke's law and the figure for Boyle's Law. According to the former, the elongation of a wire is *directly* proportional to the load, and the graph is a straight line. The latter states that the volume of a gas is *inversely* as the pressure, and the graph is not a straight line, but a curve. Mathematically, this curve is one of a class called hyperbolas. A direct proportion is represented by a straight line; a reciprocal or inverse proportion is represented by a so-called equilateral or rectangular hyperbola.

EXAMPLES

1. *The mass of a body having a given volume varies directly as the density, or $M = V \times d$. For example, equal volumes of cork, wax, wood, aluminum, iron, and lead have different masses in proportion to their respective densities. The following table shows the relation between the mass and density for a constant volume of 3 cubic centimeters, M being expressed in grams and d in grams per cubic centimeter. Complete the table and plot the graph with reference to two axes at right angles, using any convenient scale.*

If $d = 0$ $M = 0$
If $d = 1$ $M = 3$
If $d = 2$ $M =$
If $d = 3$ $M =$
If $d = 4$ $M =$
If $d = 5$ $M =$
If $d = 6$ $M =$
If $d = 7$ $M =$
If $d = 8$ $M =$

2. *Conversely, for bodies having a given mass, the volumes vary inversely as the densities, or $V = M/d$. That is, the greater the density of a body the less the volume must be for a given mass. Complete the following table, the constant mass being twelve in this case, and plot the graph.*

If $d = 0$ $V = \infty$
If $d = 1$ $V =$
If $d = 2$ $V =$
If $d = 3$ $V =$
If $d = 4$ $V = 3$
If $d = 5$ $V =$
If $d = 6$ $V =$
If $d = 7$ $V =$
If $d = 8$ $V =$
If $d = \infty$ $V =$

3. (*a*) *The loads necessary to produce a given deflection in different beams are directly proportional to the widths of the beams.*
(*b*) *Conversely, the deflections of different beams under the same load are inversely proportional to the widths of the beams, assuming, of course, that the beams are alike except in width.*
Prepare graphs that will illustrate each of these laws.

In the **indicator diagram** shown in Fig. 34, page 56, each half of the diagram shows the changes of steam pressure in one end of the cylinder for a complete or double stroke of the piston. The area inclosed within *ABCDEA*, Fig. 50, represents the work done in one end of the piston, and the corresponding area, *A′B′C′D′E′A′*, is the work done in the other end — for each square inch of piston area. The total work done by the steam for a complete stroke, forward and back, is the sum of the two multiplied by the area of the piston. In the diagram the two areas overlap in part, but this is only a matter of con-

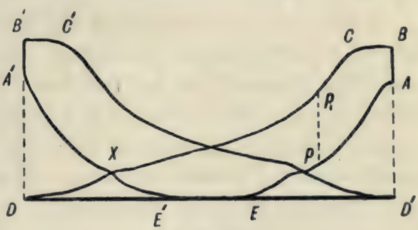


FIG. 50.

venience in the mechanical process of taking the card from the engine; in computing the work done this overlapping area is counted twice.

When the piston is at the right end of a stroke in the diagram, a small amount of steam is confined in that end of the cylinder under pressure $D'A$ to form a cushion, as explained on page 86, while the other end at that instant is in communication with the air, and hence has little or no pressure above the atmospheric. In this position, steam is admitted to the right end, and the pressure in that end instantly jumps from $D'A$ to the boiler pressure $D'B$. The piston moves to the left. The right port apparently remains open to the full boiler pressure for about one eighth of the stroke, when it closes (at point C); for the balance of the stroke the steam works by expansion, gradually falling from boiler pressure at C to atmospheric pressure at D. At D the piston completes the half stroke and starts back. The right end of the cylinder is now open to the atmosphere (having opened at X even before the forward stroke was completed) and remains so until point E is reached, when it is closed for the purpose of retaining sufficient steam to form the cushion by being compressed from E to A. If it were not for this compression, the work done by this end of the piston for a complete stroke would be represented by the somewhat triangular area $DD'BCD$. Out of this area we must subtract the area $ED'A$, lost by compression during the return stroke.

While this subtraction is mathematically correct, it does not give a correct idea of the actual transaction. The work of compression represented by the area $ED'A$ is really done by the steam in the other end of the cylinder, and properly should be subtracted from the area $D'DB'C'D'$. The two halves of the diagram should be read together. The right end actually does the full amount of work represented by the area $DD'BC$, a part of this being used up in the work of compressing the hold-over steam in the other end of the cylinder (area $E'DA'$) and the balance being given to the engine for useful work. We must not only read the conditions in the two ends of the cylinder at the same time, but also in their relations, each to the other. As

the pressure in the right end of the cylinder falls from B to D, the pressure in the other end changes from D' to B' by way of E' and A', the compression commencing at E' and the admission of live steam occurring at A'. From the intersection X of these two lines the pressure on the driving end of the piston is actually less than on the other end, and if it were not for the kinetic energy of the fly wheel the piston would not complete its stroke, but would bound back before reaching D.

Hence the area $DD'BCD$ *minus* area $E'DA'$ is the *effective* work done during a half revolution of the engine, and $D'DB'C'D'$ *minus* $ED'A$ is the effective work for the other half. Putting this in algebraic form, we have

$$(DD'BCD - E'DA') + (D'DB'C'D' - ED'A)$$

as the total work done during a complete stroke, or revolution. This can be transformed into

$$(DD'BCD - ED'A) + (D'DB'C'D' - E'DA'), \text{ or}$$
$$ABCDEA + A'B'C'D'E'A' ;$$

thus proving what was before stated, that this last result represents the actual mathematical value of the effective work done, even if it does not picture the transaction correctly.

The mean effective pressure, referred to on page 85, would be the average of an infinite number of lines drawn like pp_1 in the diagram. It is the combined areas $ABCDEA + A'B'C'D'E'A'$, divided by twice the length of stroke DD'.

CHAPTER VII

CENTER OF GRAVITY

Center of Figure. — The center of figure of a straight line is at its middle point; as much of the figure lies on one side of the point as on the other. With equal facility we can locate the center of figure of a circle, an ellipse, a parallelogram, a sphere, a cylinder, or any other symmetrical figure. It coincides with the **center of symmetry**, as defined in geometry. The center of figure of a spherical shell is the same as if it were a solid sphere; a hollow box has the same center as a solid block of the same size and shape; likewise the center of a length of pipe is not in the material of the pipe, but at the middle point of the axis of the inclosed cylindrical space.

Even if the figure is not strictly symmetrical, if it possesses some degree of regularity, there may be in it a point which may properly be called its center of figure. A triangle can be imagined to be made up of an infinite number of parallel straight lines, as in Fig. 51, and the center of each line will be in the median CD. Since all these lines taken together constitute the triangle, the center of the triangle must be somewhere in the median CD. If a second set of lines were drawn parallel to AC or BC, their centers would all lie in a second median. The intersection

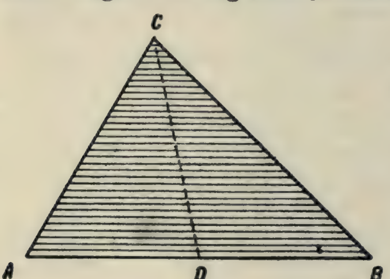


FIG. 51

of two of its medians is, therefore, the center of figure of the triangle.

Center of Mass. — If a geometrically symmetrical body is also composed of uniformly distributed particles, the center of figure is the position of a central particle. And if, furthermore, the body is of a strictly homogeneous substance or of uniform density, the central particle is also an average point about which the mass of the body is distributed — or the center of mass, as it is commonly called.

If this body, howsoever symmetrical in shape, is not of uniform density, the center of mass is not so readily located.

If a piece of lead is glued to the end of a cork, the center of mass of the combined masses is in the line connecting the centers of the lead and cork, and nearer the former in inverse proportion to the respective masses.

Center of Gravity. — According to the law of gravitation, every particle of a body participates in the action of gravity, and (if the body is small in comparison with the size of the earth, so that the lines from its different particles to the center of the earth may be regarded as parallel) the weight of the body is the sum total of these minute attractions — subject to correction for the earth's rotation, as already explained, and for other minor influences. For many purposes in mechanics, this resultant attraction may be represented as a single force, the point of application of which is called the center of gravity.

The importance of this conception is that the action of gravity, or any other force acting equally and in the same direction on all the particles of the body, is the same in effect as if the entire mass of the body were condensed in its center of gravity, the remaining particles of the body being imagined as weightless or inactive.

Since the laws of mechanics deal with masses only through their relation to forces acting upon them, there is no dis-

crimination that we shall need to exercise in using the terms "center of mass," "center of gravity," and "center of weight"; we may use one for another without danger of error.

To locate the Center of Gravity of a Body. — It has been shown that the center of figure of any symmetrical body can be found by simple geometric construction, and also that if its density is uniform, its center of mass and center of figure will coincide. If it is a hollow body, — a tube, box, ring, or spherical shell, of uniform thickness and density, — this central point is still called the center of mass, even though there is no central particle of the substance.

Having located the center of mass of each of several symmetrical bodies by geometric methods, we can use the results to prove the following

Proposition. — *If a body is supported freely and loosely on a pivot, it will adjust itself so that the center of gravity will be vertically below (or above) the point of support.*

(*a*) Prepare two pieces of cardboard, one triangular and the other a parallelogram. Locate the center of each by geometry. In each make two pin holes near the edges, and not too near each other. Prepare a plumb line by tying some small, heavy object to a fine thread. On a fine needle driven into the wall suspend the triangle from one of the pin holes, and hang the plumb line from the same support. When both have come to rest, mark the lower edge of the triangle where it crosses the plumb line. Remove from the pivot, and draw a line from the point of support to the point marked on the edge. Now suspend the triangle from the second pin hole, and determine a second line. Do these lines intersect at the center of figure as determined by construction? If not, determine the amount of error by measuring the distance between the two locations. When the triangle was supported on the needle, what was the position of the center of gravity relatively to the point of support?

In the same manner hang the parallelogram on the needle from each of the pin holes in succession, and note results.

(*b*) Remove the cover and bottom of a pasteboard box. Attach a thread to several different points in the edge of the cardboard, after the manner in which the objects were suspended in the preceding case, and see whether the plumb line passes through the center of the box.

These experiments will suffice to show that the center of gravity of a body ordinarily assumes a position vertically *below* the point of support. This is a simple fact that hardly needed demonstration; it is in perfect accord with an endless number of phenomena that we observe from day to day.

To support a body from a single point with its center of gravity vertically *above* the point of support — such as a cone balanced on its apex, or a parallelopiped on an edge — is theoretically possible, but in practice the most skillful equilibrist would require some small area — more than a mathematical line or point — for a base of support.*

Equilibrium. — (*a*) *When a body is supported from a pivot, with its center of gravity vertically above or below the point of support, it is said to be in a position of equilibrium.* The weight of the body, acting like a single downward force exerted at the center of gravity, is balanced or equilibrated by the resistance of the pivot. If the center of gravity were not in vertical line with the point of support, as in Fig. 52, the pivot could not furnish a resistance equal and opposite to the force exerted at C by the weight of the body. To assume a position of equilibrium the body must rotate around the pivot until the vertical CD passes through P.

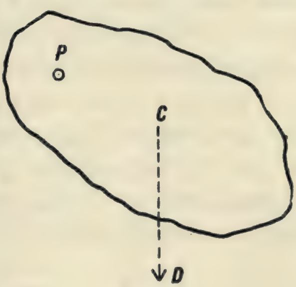


FIG. 52

* The center of gravity of a captive balloon is above the point of support; likewise for the float on a fishing line, but in such cases the action of gravity is complicated with the phenomenon of buoyancy.

(*b*) Problems of equilibrium are more frequently met with in *bodies resting on a base of support;* pivoted bodies came first in order of consideration, not because of their greater importance, but because it is simpler to deal with a single point of support than with an area of support. If a rectangular block rests on one of its faces on a horizontal plane

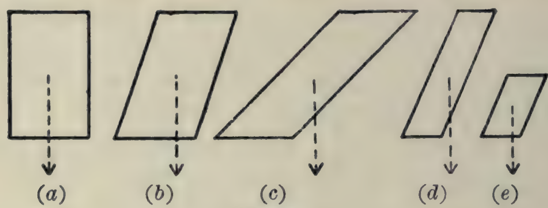


Fig. 53

(Fig. 53 *a*), it is in equilibrium, because the vertical drawn through the center of gravity passes through the base of support, and the force of gravity is balanced by the resistance of the object supporting the base. If the block were not rectangular, as in Figs. 53 *b*, *c*, *d*, and *e*, it would be in equilibrium or not, depending upon the area of base, the vertical height, and the acuteness of the angles. Figures *b* and *c* have the same area of base and the same vertical height, but the angles of *c* are such that the vertical from the center of gravity falls outside the base of support and the body is not in equilibrium. Figures *b* and *d* have the same angles and the same vertical height, but Fig. *d* is not in equilibrium because of its smaller base. Figures *c* and *d* would each be toppled over by its own weight.

Fig. 54

A body in equilibrium on a horizontal plane might not be in equilibrium if the plane were inclined, as shown in Fig. 54. A sphere is always at equilibrium on a horizontal plane, but is never so on an inclined plane.

If a body rests on several isolated points, as a surveyor's

tripod, or a table, the base of support is the convex polygon that would be determined by winding a thread around the external points of support. There may be many other supporting points within the perimeter thus determined; but as they do not extend the area of support, they do not add to the stability of the body as regards overturning.

Stability of Equilibrium — Stable; Unstable; Neutral. — (*a*) *Pivoted Bodies.* It has been stated that when a body is suspended freely from a pivot its center of gravity assumes a position vertically below or above the point of support; otherwise the body is not in equilibrium. One exception, however, should be noted. If the point of support coincides with the center of gravity, the body is at equilibrium in any position, and will remain at rest wherever placed by turning it around the pivot.

When the center of gravity of a pivoted body is vertically above the point of support its position is very insecure; any slight displacement of the body will cause it to roll over with its center of gravity downward. Though the body was in equilibrium it lacked stability, or was in a position of **unstable** equilibrium.

But the position it naturally assumes, with the center of gravity vertically below the point of support, is the one of maximum stability. If it is displaced from this position, it returns to it by force of its own weight. It is in **stable** equilibrium.

When the point of support coincides with the center of gravity, the equilibrium of the body is **neutral**.

From these considerations, therefore, we derive two tests by which to judge the **kind of equilibrium** of a pivoted body.

(i) Its equilibrium is stable if the body tends to return to the same position after a slight displacement; but if a slight jar or disturbance causes the body to move still farther from its first position, its equilibrium was unstable; if it remains

wherever it is placed, its equilibrium is undisturbed by the change and is said to be neutral.

(ii) When the center of gravity is in its lowest possible position (vertically below the point of support), the body is in stable equilibrium; when the center of gravity is at its highest possible position (vertically above the point of support), the body is in unstable equilibrium; when the center of gravity coincides with the point of support, the equilibrium is neutral.

A third test of the stability of a body can be deduced from the consideration that any motion of the body around the pivot, which results in elevating the center of gravity, requires the same amount of work — the same expenditure of energy — as if the whole mass were raised bodily through the same distance. If the rod shown in Fig. 55 weighs 8 pounds and its center of gravity is raised through a vertical height of 2 feet by rotating through the arc CC', then the work done is 16 foot-pounds, although a part of the rod has not been raised at all.

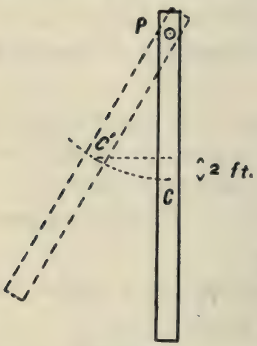


FIG. 55

(iii) Now consider the three positions of the center of gravity relatively to the point of support — vertically below; vertically above; and coinciding with it — and the performance of work involved in any slight displacement from each of these positions of equilibrium. In the first case the center of gravity is raised and hence work is done *upon the body* to displace it from a position of stable equilibrium. In the second case the center of gravity is lowered, and hence when the body is displaced from a position of unstable equilibrium work is done *by the body*, its potential energy being converted into energy of motion. In

the third case the displacement of the body is not accompanied by any change in the position of the center of gravity, and hence a body in neutral equilibrium has the same potential energy in all positions.

(*b*) *Bodies resting on a Base.* A right circular cone placed in various positions on a horizontal plane will afford typical illustrations of the three kinds of equilibrium. Resting on its base it is stable; balanced on its apex it is unstable; and lying on its side, on an element, it is in neutral equilibrium.

In general, if a body rests (at equilibrium, of course) on a base of any appreciable area, it is stable, because any displacement tending to overturn it, or give it a new base of support, will result in elevating the center of gravity, and hence require an expenditure of energy. If it rests on a point or line, as a cube on a corner or an edge, with its center of gravity vertically above, in such manner that any displacement of overturning would lower the center of gravity, it is unstable. But if it rests on a point or line in such manner that the support may be shifted to other points or lines — a sphere, or a cylinder on its side — without raising or lowering the center of gravity, it is in neutral equilibrium.

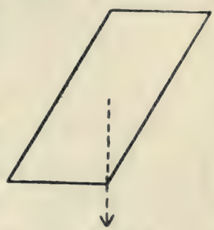


Fig. 56

If a body rests on an area of support with the center of gravity vertically above any point in the perimeter of the base, as in Fig. 56, we have a limiting case. Any slight displacement to the right will prove the present position of the body to be one of unstable equilibrium, while from a displacement to the left it will recover its present place as if it were stable. Such instances are sometimes called cases of mixed equilibrium.

EXAMPLES

1. *If a rectangular block rests with one face on a horizontal plane, what kind of equilibrium does it possess? Why? How must it be*

placed to be in unstable equilibrium? Could it be placed in a position of neutral equilibrium? Why?

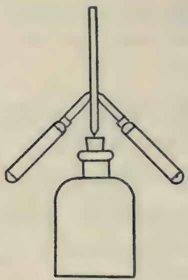


FIG. 57

2. *How must a right circular cylinder be placed to be in stable equilibrium? Could it be placed in a position of unstable equilibrium?*

3. *Can a sphere be placed in a position of stable equilibrium on a horizontal plane?*

4. *A pencil cannot be readily balanced on its sharpened point, but if a knife blade be stuck into each side, it can then be balanced on its point from an elevated support, as in Fig. 57. Explain.*

5. *Fig. 53 e is stable, while Fig. 53 d, having the same base, is not stable. Explain.*

Degree of Stability. — We have observed that a pivoted body has a choice of but three positions of equilibrium, and in each of these positions we regarded it as possessing a certain *kind* of equilibrium. These differences of kind we expressed by means of adjectives which signified merely the stability or non-stability of the body. In algebraic notation we could have said that in one position the body is positively stable to a certain degree, and in another position it is *un*stable or negatively stable. The numerical value of this degree of stability may be great or small, depending upon the weight of the body, and the distance between its center of gravity and the point of support. In a position of neutral equilibrium this distance vanishes and the stability is zero. It follows, therefore, that the equilibrium of a pivoted body may be represented in all its aspects by a single algebraic quantity, of which the numerical value will represent the degree of stability, and the sign of quantity will show the *kind* of equilibrium.

When a body rests on a base of support its degree of stability is measured by the work necessary to overturn it. For example, let us investigate the stability of a rectangular

block resting on a horizontal plane. Suppose it measures 2 feet × 3 feet × 4 feet and weighs 150 pounds. There are six cases to be considered, since the work to be done in overturning depends upon which face the block rests upon, and the edge over which it is turned.

These cases are as follows:

1. Resting on 2 × 3 base.
   (*a*) Turned over 2-foot edge.
   (*b*) Turned over 3-foot edge.
2. Resting on 3 × 4 face.
   (*a*) Turned over 3-foot edge.
   (*b*) Turned over 4-foot edge.
3. Resting on 2 × 4 face.
   (*a*) Turned over 2-foot edge.
   (*b*) Turned over 4-foot edge.

SOLUTION

1. Resting upon a 2 × 3 face.
   (*a*) Turned on 2-foot edge.

If the body starts from the initial position $ABCD$ (Fig. 58) and is overturned on the 2-foot edge, its center of gravity will describe the arc $OO'O''$. Since the edge AB is 3 feet and the height BC is 4 feet, the distance OA will be $\frac{\sqrt{3^2+4^2}}{2}$, or 2.5. When the body

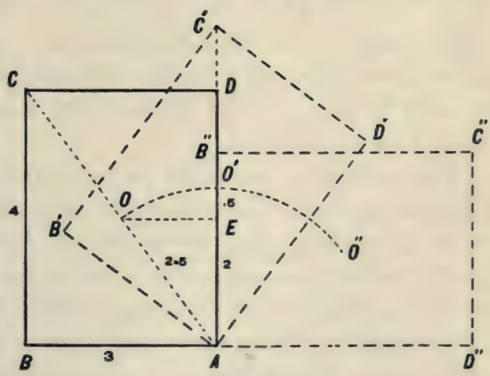


FIG. 58

reaches the position $AB'C'D'$, it occupies a position of unstable equilibrium and no further work will be required for

overturning; if it is displaced to the slightest degree beyond this position, it will then fall by its own weight to the position $AB''C''D''$. Therefore, the work necessary to overturn the body from its first position of stable equilibrium is the amount required to raise the center of gravity from O to O', which, as we have already learned, is equivalent to raising the entire mass through the same distance. In describing the arc OO' the center of gravity is raised through the vertical distance EO', equal to 0.5 foot. The weight of the body being 150 pounds, the work done is 150×0.5, or 75 foot-pounds.

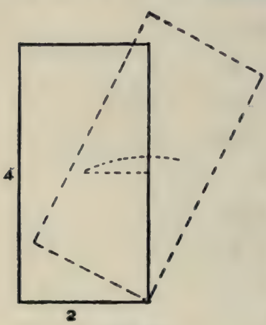


FIG. 59

This gives us a measure of the stability of the body as opposing any effort to turn it from the given position, over the edge specified. It would be hardly sufficient, however, merely to say that the stability of the body is 75 foot-pounds; we must specify all incidental conditions upon which this depends, — the face upon which it rests, and the edge over which it is turned.

(*b*) Resting on the same base and turned on the 3-foot edge the vertical distance through which the center of gravity will be raised (Fig. 59) is 0.236 foot, and the work done is 35.4 foot-pounds.

The remaining cases are to be solved by the student and the results entered in the upper table on opposite page.

By comparing these results it will be seen that the body possesses the greatest stability when the center of gravity is as low as possible and the base as wide as possible.

This will be shown still more clearly if the results are rearranged in proper places in the lower table on the following page, in which the width of base *in*creases in spaces from left to right, and height of center of gravity *de*creases from the top downward.

		Height of Center of Gravity in Feet	Width of Base in Feet	Stability
I	*a*	2	3	75
	b	2	2	35.4
II	*a*			
	b			
III	*a*			
	b			

For any given base the stability is greater when the center of gravity is lower; for any given height of the center of gravity, the stability is greater when the base is broader.

Height of Center of Gravity		Width of Base		
		2 Feet	3 Feet	4 Feet
	2 Feet	35.4	75	
	1.5 Feet			
	1 Foot			

EXAMPLES

1. *What work is done in overturning a cube of metal measuring 1 foot each way and weighing 500 pounds?*

2. (*a*) *A mass of metal measuring 1 foot × 1 foot × 2 feet and weighing 1000 pounds rests on one end. What work is done in overturning it?*

(*b*) *What work will be done in overturning it from a position of rest on one side, turning it on the 2-foot edge?*

(*c*) *What if turned from the same position on the 1-foot edge?*

3. *A stove 3 feet high, 2 feet wide, and 3 feet long, weighing 300 pounds, rests on legs 6 inches high. Another, weighing 500 pounds, rests on legs 12 inches high and measures 3 feet high, 3 feet long, and 3 feet wide. Which stove is the more stable, and in what ratio?*

NOTE. Disregard weight of legs and consider center of gravity as being at center of figure of the "body" of stove.

CHAPTER VIII

PRINCIPLES OF MACHINES. THE LEVER

Tools and Machines. — By means of a few hand tools a blacksmith overcomes the resistance of a piece of iron and forges it into almost any desired shape, in a manner that would not be possible by unaided human effort. Similar results are accomplished in any material, as wood or stone, by means of suitable tools, or with greater facility by woodworking and stoneworking machinery operated by power. Most tools and machines of this type, used for purposes of construction and including many of the highly specialized devices used in various factories, serve the purpose of cutting and shaping materials.

Under a different head we could classify machines that are used for lifting loads or moving masses of any kind. This class would include chain hoists, cranes, pumps, air blowers, elevators, etc., and in the same list we might even include a dynamo, in which the resistance overcome is something more than the mere mass of the moving parts and the results obtained are not so readily discerned.

A third class would be machines that are designed to accomplish delicacy and accuracy of motion, without overcoming any great resistance, and would include weaving machines, sewing machines, typewriters, and typesetting machines.

This is by no means a complete list or classification of the many different kinds of tools and machines in use, but it is sufficient for the purpose of opening the way to a definition that will indicate the fundamental character or essential nature of a machine.

A machine is an instrument by which a given force is made to accomplish indirectly a result that would not be possible by direct application of the force without the intervention of such a medium. *It is a device that serves to modify a force or motion*, in magnitude or direction, or in both respects. Any tool, implement, device, contrivance, instrument, appliance, or apparatus, by which this is accomplished, directly or incidentally, involves a mechanical principle and is a machine, according to our definition. Animal motions, even, are due to the various mechanisms of which the anatomical structure is composed.

Efficiency of Machines. — As already stated, many machines are intended to lift loads and in other ways overcome resistance. But even those which are designed to accomplish merely a delicate movement of any part in a certain direction cannot be operated except by the application of energy supplied from some source external to the machine. If the parts of the machine could move without frictional resistance, it would be capable of doing an amount of useful work exactly equal to the energy applied. But this is practically impossible; owing to the frictional resistance, air currents, etc., due to the moving parts of the machine, some of the applied energy is dissipated, or frittered away, in the form of useless motion, heat, sound, and at times even light and electricity. If a machine performs only 385 foot-pounds per second of useful work at the expense of one horse power of energy (550 foot-pounds per second) applied to it, its efficiency is only 70 per cent. The efficiency of a machine depends upon its structural features and the condition of the bearing parts, and is calculated as the ratio of "energy output" to "energy input" in a given time.

A complex machine may be more efficient in some parts than in others. A machine-shop planer, for instance, might

be conveniently segregated into three parts: (1) the horizontal bed; (2) the driving pulleys and accessory parts conveying motion to the bed; and (3) the countershafting. The countershafting may convey to the machine proper 96 per cent of the power taken from the main shaft; the driving parts of the planer may furnish to the bed only 82 per cent of the power received from the countershaft; and of this the losses due to motion of the bed may leave only 88 per cent. Hence the complete machine would furnish in useful work only 88 per cent of 82 per cent of 96 per cent of the power taken from the main shaft; or its efficiency would be less than 70 per cent.

The Simple Machines. — Any mechanical contrivance, howsoever complicated, can be analyzed into certain elementary parts, commonly designated as the **mechanical powers**, or **simple machines**. These are:

1. The lever (including the bent lever and bell crank).
2. The wheel and axle (sometimes classed as a lever).
3. Pulleys.
4. Inclined plane.
5. The wedge (which may be included under the inclined plane).
6. The screw (which may also be included under the inclined plane).
7. The toggle joint (sometimes called by analogy the "knee joint," or "elbow joint").

All the elementary or component parts of any machine could be classified under these heads. The most complicated machine is nothing more than an assemblage of parts involving combinations and modifications of these elementary mechanisms.

The Lever. — A lever is a rigid rod free to move about a single point or bearing called the fulcrum. It is the sim-

plest form of a machine, and from it we can deduce the **fundamental principles** and considerations **of all the mechanical powers.**

(I) **Principle of Virtual Work, or Virtual Velocities.** — If a weight W (Fig. 60), suspended from one end of a rigid rod, is balanced by a force P acting vertically downward on the other end, or on any point beyond the fulcrum, the relative values of P and W necessary for equilibrium depend upon the distances a and b, or AF and BF.

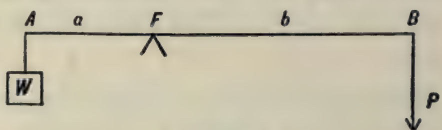


FIG. 60

The two forces are inversely proportional to the perpendicular distances from their lines of action to the fulcrum.

$$\frac{P}{W}=\frac{a}{b}. \tag{9}$$

For example, if the short arm of the lever is 3 feet long and the other arm 5 feet, a weight of 10 pounds suspended from the end of the short arm would be balanced by a force of $10 \times \frac{3}{5}$, or 6 pounds, applied vertically downward at the other end of the rod.

This can be proved by the principle of virtual work. Assuming that the rod is without weight and that the forces P and W produce equilibrium, suppose that the rod be displaced through an angle γ (Fig. 61) in either direction around the fulcrum. If the weight is raised to a position W', through a vertical height h, the other end at the same time falls through a vertical distance k. From similar triangles $\frac{h}{k}=\frac{a}{b}\cdot$ If $h=1$ foot, k is therefore equal to $\frac{5}{3}$ feet. Any displacement of the rod which raises the weight one

foot requires an expenditure of 10 foot-pounds of energy. But, as we have shown, this displacement of the weight would be accompanied by a falling of the other end, or a movement of the applied force, through a distance $\frac{5}{3}$ feet, and since the applied force is 6 pounds, the energy furnished by it is $6 \times \frac{5}{3}$, or 10 foot-pounds. The work done by the applied force is therefore just equal to the work done upon the weight, and on the whole there is no gain or loss of energy. This is strictly in accord with the law of the conservation of energy, and is applicable as a test of the equilibrium of any of the mechanical powers. That is, to determine whether the applied force is just sufficient to balance the weight or resistance, we imagine a slight displacement of the machine in such manner as it is free to move, and then calculate the product of the weight times the vertical distance through which it moves. This should be equal to the product of the applied force times its vertical distance; or, referring to the figure, we should have

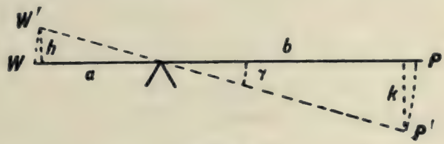


FIG. 61

$$Wh = Pk. \tag{10}$$

This is called the **principle of virtual work**, or **virtual velocities**, and was first enunciated by Stevinus and Galileo about 1600. The adjective "virtual" is used to signify that there is no real motion or displacement, and no work is actually performed either upon the weight or by the applied force; but the result attained by the supposition that motion does take place, and that work is done, is virtually the same as if the machine really moved, in spite of the fact that it is in equilibrium.

If $Wh = Pk$, then $\frac{W}{P} = \frac{k}{h}$. Since $\frac{k}{h} = \frac{b}{a}$, we have $\frac{W}{P} = \frac{b}{a}$, thus proving our proposition, that two balancing forces applied to a lever are inversely proportional to the distances of their points of application from the fulcrum.

(II) **Principle of Moments.** — In the case under consideration, where the two forces act at right angles to the lever and on opposite sides of the fulcrum, the distances a and b are called the lever arms. The product of a force times its leverage, or lever arm, is called the **moment of the force.** A force acting at the end of an arm 2 feet long has twice the advantage of the same force if its lever arm is only one foot.

The principle of moments asserts that two forces acting on a lever are in equilibrium if their moments are equal and tend to turn the lever in opposite directions. The writings of Archimedes, who lived about 250 B.C., show that he was familiar with this principle. Its demonstration requires only a simple transformation of the expression $\frac{W}{P} = \frac{b}{a}$ to the form $Wa = Pb$. In the case under consideration (Fig. 60) the weight, 10 pounds, with a leverage of 3 feet, is exactly balanced by the opposing force of 6 pounds with a leverage of 5 feet, because both have moments of 30 units.

A moment is usually designated as so many pound-feet, or ton-feet, for obvious reasons. These should not be confounded with the unit of work, the foot-pound. In both cases we use a compound word made up of a unit of force and a unit of distance, but the order of combining the component words is different in the two cases, and furthermore the unit of distance represents a leverage in one case while in the other case it refers to the distance through which a weight is raised or some other resistance overcome.

If several forces and weights act on the same lever simultaneously, the sum of the moments of the applied forces

must be equal and opposite in direction to the sum of the moments of the weights, to produce equilibrium. And if any of the forces act in a direction tending to aid the weights,

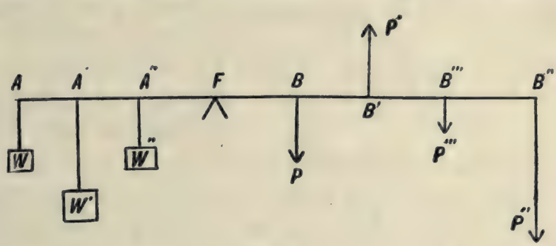


FIG. 62

such forces should be given a negative moment. In Fig. 62 the forces and weights are in equilibrium, provided

$$W \times AF + W' \times A'F + W'' \times A''F =$$
$$P \times BF - P' \times B'F + P'' \times B''F + P''' \times B'''F.$$

Moment due to Weight of Lever. — A rod of uniform dimensions and density supported midway between the two ends is in equilibrium, because its center of gravity is in vertical line with the point of support. But if the fulcrum is at some point other than the center, as illustrated in Fig. 63, the weight of the rod acts like a load applied at its center of gravity; as if the entire weight W of the rod were concentrated at that point. Hence, if such a rod were used for a lever, we could allow for its weight by considering its effect equal to moment $W \times CF$. If the center of gravity is directly over the fulcrum, this moment is zero and the weight of the rod may be disregarded.

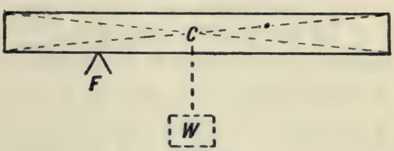


FIG. 63

Sometimes it is easy to allow for the weight of the rod by finding the proportion of weight on each side of the fulcrum, and multiplying the weight of each part by the distance of its center

from the fulcrum, as illustrated in Fig. 64, in which W_1 and W_2 are the weights of the portions of the lever on opposite sides of the fulcrum. If the weight of each part is assumed to be massed at its center, then the moment of the heavy rod is equal to that of a weightless rod with the weights W_1 and W_2 suspended as shown.

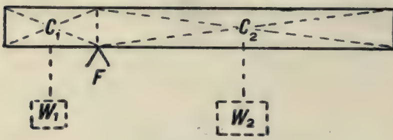


FIG. 64

This method seems to suggest itself to most persons more readily than the more direct method previously described, but it is a roundabout procedure and sometimes involves difficulties. The method illustrated in Fig. 63 is simpler and better.

EXAMPLES

1. *A heavy rod is balanced at its middle point and then a weight of 40 pounds suspended from a point 2 feet from the fulcrum.*

(*a*) *At what distance on the other side of the fulcrum must an 18-pound weight be placed in order to produce equilibrium?*

(*b*) *What weight would have been sufficient to produce equilibrium if it had been placed 3.5 feet from the fulcrum?*

2. *A heavy rod 3 feet long is balanced on a fulcrum at its middle point. If a weight of 10 pounds is suspended from one end and a second weight of 5 pounds is placed at a point one foot from the fulcrum on the same side, what downward pull must be exerted on the other end of the rod to produce equilibrium? What if the 5-pound weight had been placed one foot from the fulcrum on the side toward the downward pull?*

3. *If a stick of timber 9 feet long and weighing 16 pounds is supported on a fulcrum 3 feet from one end, what weight must be suspended from this end to produce equilibrium? Where would a 12-pound weight be placed to produce the same result?*

4. *A uniform iron rod 11 feet long and weighing 40 pounds is supported on a fulcrum 3.5 feet from one end. If a 14-pound weight is hung from this end, where must a second weight of 28 pounds be placed to produce equilibrium?*

5. *For the principle of work we found that $Wh = Pk$, and for the principle of moments $Wa = Pb$. Explain the difference between these principles by contrasting the meanings of h and k with a and b.*

Three Kinds of Levers. — The applied force and the load acting on a lever are not always on opposite sides of the fulcrum. If a rod is fixed at one end, as shown in Figs. 65 and 66, a weight acting at any point may be balanced by a force exerted at any other point.

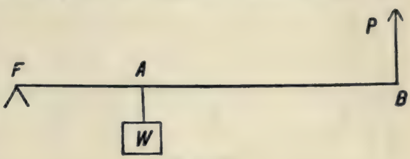


FIG. 65

The relative values of P* and W* depend upon the distances AF and BF, exactly as determined for the lever of the first kind, from which we deduced the principle of moments. Whatever the relative positions of P, W, and F, these principles apply with equal strictness. In Fig. 65, since $W \times AF = P \times BF$, and BF is greater than AF, the applied force is less than the weight. In Fig. 66 it is greater.

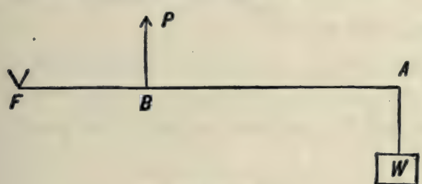


FIG. 66

If P and W are on opposite sides of the fulcrum, the lever is said to be of the "first order." Figure 65 illustrates a lever of the "second order," and Fig. 66 one of the "third order."

In levers of the first order the applied force may be greater

* The employment of the letters P and W, to represent respectively the applied force and the weight, has become a matter of conventional usage. It is assumed, of course, that W does not always stand for a weight or gravitational action, but may be a load, or a tension, or resistance of any kind whatsoever. The applied force P is frequently called the "*power*"; whence the symbol P. This use of the word "power," referring to a force merely, is not consistent with our previous acceptance of its meaning in the sense of "rate at which a machine can do work." For the sake of accuracy we shall adhere to the expression "applied force," symbolized by the capital P.

or less than the load. In levers of the second order the applied force is always less than the load, while in levers of the third order the load is less.

Compound Levers. — In some devices — certain gate latches and wagon brakes, for examples — a train of levers is used to multiply the force or motion, as the case may be. As shown in Fig. 67, the weight, or resistance, of one is propagated to the next as an applied force. It is not necessary that all in the series be levers of the same order.

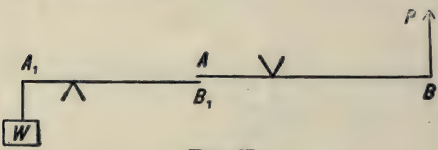


FIG. 67

Other examples of compound levers are found in platform scales, typewriting machines, piano keys, the trigger of a gun, and in railroad switches.

EXAMPLES

1. *To which class of levers would you assign each of the following devices:*

Pair of pliers, sugar tongs, scissors, nut cracker, blacksmith's vise, blacksmith's tongs.

2. *Name five devices in which the lever is used. At least one of the five must refer to the lever of the third order.*

3. *How many of the three kinds of levers are illustrated in the ordinary use of a crowbar?*

Safety Valves. — The ordinary safety valve, used as a boiler attachment to prevent explosions, is a lever of the third order. A valve V (Fig. 68), in direct communication with the interior of the boiler, is pivoted at B to a bar AF. Whenever the upward pressure on V becomes sufficiently great to over-

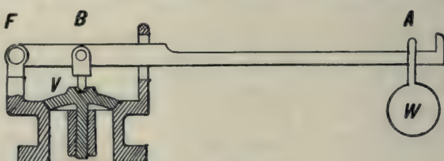


FIG. 68

come the resistance due to the weight of W and of other parts of the mechanism, the lever and valve are moved upward, turning on F as a fulcrum. By this means a part of the contents of the boiler is allowed to escape, and when the boiler pressure is sufficiently reduced the valve drops back into its seat.

Unless the area of the valve is sufficient to allow the steam to escape as fast as it is generated, perfect safety is not secured. As the generating capacity of the boiler depends mainly upon the area of the grate surface, it is customary to allow one square inch of valve area for every two square feet of grate surface.

EXAMPLES

1. *How would you compute the total upward pressure on the valve from the gauge pressure? Should allowance be made for the atmospheric pressure on the top of the valve?*

NOTE. — The reading of a steam gauge indicates, not the actual boiler pressure, but the difference between this and the pressure of the atmosphere.

2. *What pressure per square inch (by gauge) would be necessary to raise a safety valve constructed and adjusted as follows:*

Weight of valve, 8 pounds.
Diameter of valve, 3 inches.
Valve pivoted 4 inches from fulcrum.
Weight of lever 12 pounds (uniform rod).
Total length of lever, 2 feet.
Weight of 150 pounds suspended from extreme end of lever.
All as illustrated by diagram, Fig. 69.

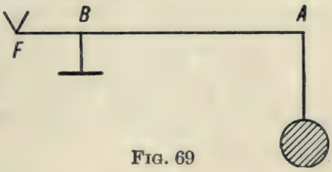


FIG. 69

3. *At what point on this lever arm would it be necessary to place a 200-pound weight to withstand the same pressure?*

4. *At what point must the 150-pound weight be placed to balance a boiler pressure of 100 pounds per square inch (by gauge)?*

5. *What weight must be placed at the end of the rod to balance a boiler pressure of 90 pounds per square inch (by gauge)?*

6. *If a boiler is over a grate 5 feet square, what diameter of safety valve should it have?*

Pressure on the Fulcrum: Parallel Forces. — (*a*) *Levers of the First Order.* If a ball weighing 10 pounds (Fig. 70) is fastened to one end of a weightless lever, 3 feet from the fulcrum, it can be balanced by a second ball weighing 6 pounds and fastened on the other end at a distance of 5 feet from the fulcrum. It is obvious, in this instance, that the fulcrum bears the entire load of 16 pounds, $W + P$. And since the entire system of combined masses is at equilibrium, its center of gravity must be in vertical line with the point of support, as if the entire weight (16 pounds) were acting downward on that point.

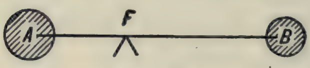


FIG. 70

Since the distances of A and B from the fulcrum are inversely as the weights acting at those points, it follows as a general **principle** that if two **parallel forces** in the same direction act on two different points of a body, their resultant is equivalent to a single force equal to their sum and acting at a third point whose distances from the two points of application are inversely as the forces.

If more than two parallel forces are acting simultaneously on the same body, the resultant of any two may first be found, and this may then be combined with a third force, etc.

EXAMPLES

1. *A rectangular block of wood 1 centimeter × 1 centimeter × 2.5 centimeters is glued to a cube of lead 1 centimeter × 1 centimeter × 1 centimeter, in the manner shown in Fig. 71. Locate the center of gravity of the combined masses. (Specific gravity of lead = 11.3; specific gravity of wood = 0.6.)*

2. *A $\frac{1}{2}$-inch bolt has a head $\frac{7}{8}$ inch square and $\frac{7}{16}$ inch high. If the shank of the bolt is 2 inches long, locate the center of gravity of the entire bolt.*

(*b*) *Levers of the Second Order.* In a lever of the second order the pressure on the fulcrum is $W - P$. For example,

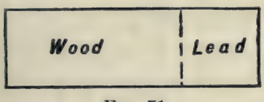


FIG. 71

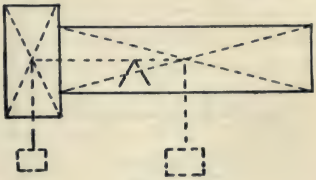


FIG. 72

if a weight of 12 pounds (Fig. 73) is suspended at a distance of 1 foot from the fulcrum, it can be balanced by a force of 4 pounds if the latter is applied at a distance of $BF = 3$ feet. In this case the fulcrum supports a load of 8 pounds.

To make this clear, imagine that BF is a stick supported by two persons, each having an end of the stick resting on his shoulder, and the 12-pound weight being suspended as stated, one foot from one end and two feet from the other. The person holding the end B will support 4 pounds, leaving the person at F to hold 8 pounds. Because, if the person at B should raise his shoulder, the lever would turn about F as a fulcrum; and as the point B would describe an arc with radius FB, while A moves with radius $FA = \frac{FB}{3}$, then by the principle of work the end B would be lifted upward by a force $P = \frac{W}{3}$, or 4 pounds. If, on the contrary, the person at F should raise his end of the rod, moving it about B as a fulcrum, the arcs described by F and A would have radii of 3 feet and 2 feet, respectively, whence the upward force exerted at F would be $\frac{2}{3} \times W$, or 8 pounds.

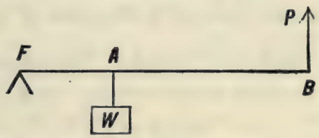


FIG. 73

If we choose to apply the principle of parallel forces (p. 128), we can say that whatever the resistance which the

fulcrum is required to exert it is equivalent to a force P' (Fig. 74) acting vertically upward. Then if the forces acting on the rod are in equilibrium, the resultant of P and P' must be equivalent to the single force of 12 pounds acting at point A — a force numerically equal to W but opposite in direction. Also, the moments of P and P' about A must be equal and opposite, or $P' \times 1 = 4 \times 2$, whence $P' = 8$.

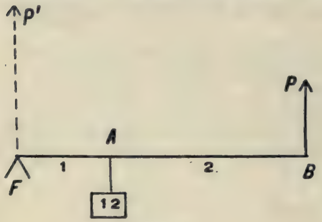


FIG. 74

(*c*) *Levers of the Third Order.*

EXERCISE

Prove that in levers of the third order the pressure on the fulcrum is exerted in the direction of the applied force, and is equal to $P - W$.

Parallel Forces. The Couple. — As a rule, but not always, two parallel forces acting on a body at different points can be counterbalanced by a third force. If the two parallel forces have the same direction, as P and P_1 in Fig. 75, the resultant is a force R equal to their sum and acting at a point F such that $BF : AF :: P_1 : P$. This case is illustrated in levers of the first order, as was fully explained on page 128. If P and P_1 are equal, F is midway between A and B, as exemplified in a beam balance having equal arms.

If the two forces are parallel and in *opposite* directions, as in Fig. 76, the resultant is a force equal to the difference between

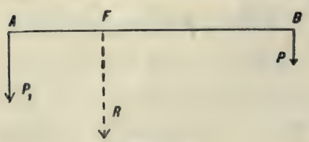


FIG. 75

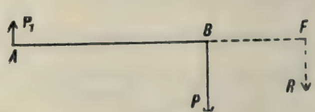


FIG. 76

the two and acting at a point F, in the line AB continued, such that $AF : BF :: P : P_1$. (Why is R in Fig. 76 drawn downward, in direction of P, and not in the opposite direction, like P_1?) If P_1

had been greater than P, the point F would have been on the side beyond A — in BA continued instead of AB. This applies to levers of the second and third orders, pages 129 and 130.

If P and P_1 are **equal** and **opposite** they have no real resultant. Two such forces acting on a body at different points constitute what is called a **couple.** No single force howsoever applied could exactly counterbalance them, and hence they have no resultant. This will become evident if we observe the changes that take place in Fig. 76, as we substitute different values for P and P_1, finally making them become equal to each other. From the relation $AF : BF :: P : P_1$ it will be seen that, if P_1 in Fig. 76 should become smaller or P become larger, the point F will in either event approach nearer to B. And conversely, if P becomes less, relatively to P_1, — that is, if P and P_1 become more nearly equal to each other, — the point F will recede from B. A few simple computations will show that when P_1 finally becomes equal to P, the point F recedes to an infinite distance. For that purpose let us substitute a few assumed values in the expression

$$\frac{AF}{BF} = \frac{P}{P_1}.$$

If $P = 2\,P_1$,	$AF = 2\,BF$,	or $BF = AB$*
If $P = 3\,P_1$,	$AF = 3\,BF$,	or $BF = 0.5\,AB$
If $P = 5\,P_1$,	$AF = 5\,BF$,	or $BF = 0.25\,AB$
If $P = 10\,P_1$,	$AF = 10\,BF$,	or $BF = 0.11\,AB$
If $P = 100000\,P_1$,	$AF = 100000\,BF$,	or $BF = 0.00001\,AB$
If $P = 1.5\,P_1$,	$AF = 1.5\,BF$,	or $BF = 2\,AB$
If $P = 1.1\,P_1$,	$AF = 1.1\,BF$,	or $BF = 10\,AB$
If $P = 1.01\,P_1$,	$AF = 1.01\,BF$,	or $BF = 100\,AB$
If $P = 1.00001\,P_1$	$AF = 1.00001\,BF$,	or $BF = 100000\,AB$

If the fractional difference between P and P_1 becomes infinitesimally small, then BF becomes infinitely greater than AB.

In some branches of applied mechanics the idea of the couple is frequently met with, although we shall not need to use it in

* Combining $AF = 2\,BF$ with $AF = AB + BF$ (Fig. 76), we get $2\,BF = AB + BF$, or $BF = AB$. Solve the other cases in the same manner.

developing our subject from an elementary standpoint. It is illustrated in turning the handles of a copying-press, if equal forces are exerted on the two ends; likewise in cutting a thread on a pipe or bolt.

On account of its peculiar mathematical conditions the couple possesses some very characteristic and striking properties. The

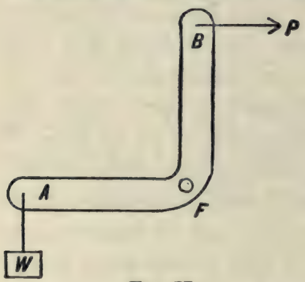


FIG. 77

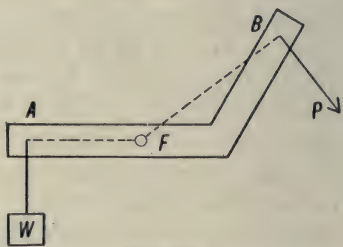


FIG. 78

sum of the moments of the two forces involved in a given couple is a constant quantity, being the same with reference to any point that may be selected in the plane of the couple, whether within or without the body acted upon by the couple; and if the body be pivoted so as to move around any point under the influence of the couple, the latter will cause no pressure whatsoever on the pivot or fulcrum.

Bent Levers. — If AFB (Figs. 77 or 78) is a rigid body free to turn about F, the forces P and W will be in equilibrium if $W \times AF = P \times BF$. No matter what the shape of the lever, nor where the fulcrum is situated, if PB and WA are perpendicular to BF and AF respectively, the principle of moments applies exactly as if AF and BF were in the same straight line.

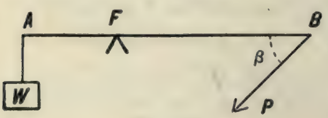


FIG. 79

Moment of a Force acting obliquely on a Lever. — Referring to Fig. 79, the moment of P with reference to the fulcrum F is not $P \times BF$. By resolving P into two components, p_1 and p_0 (Fig. 80), perpendicular and parallel to

BF, it becomes apparent that the component p_0 cannot have any effect in turning the lever; it simply pushes the lever in the direction BF against the bearings at F, without producing rotation. The component p_1, at right angles to FB, is alone effective in balancing W. Whence, if the lever is in equilibrium $W \times AF = p_1 \times BF$.

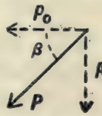


FIG. 80

If the angle PBF is called β, then the component $p_1 = P \sin \beta$, whence $W \times AF = P \sin \beta \times BF$.

Instead of resolving the force P into two components, we might have turned to the idea of the bent lever by projecting the arm BF into a direction at right angles to P. The force P acting at an angle β with a lever arm BF has the same effective moment as if it were acting at right angles to the arm B_1F, in Fig. 81. For, since $B_1F = BF \sin \beta$, it must follow that $W \times AF = P \times BF \sin \beta$. Therefore, when any force acts obliquely upon a lever, the effective moment of the force may be found in either of two ways: (1) by resolving the force into two components perpendicular and parallel to the lever arm; or (2) by projecting the lever arm into a direction perpendicular to the force.* In either case the effective moment of the force is $P \times BF \times \sin \beta$.

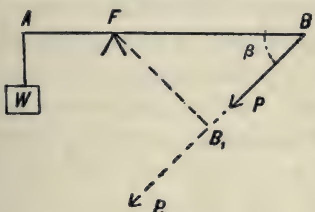


FIG. 81

It should be noted that the forces producing equilibrium in a bent lever, and likewise oblique forces acting on a straight lever, will give rise to a pressure on the fulcrum that is not merely the sum or difference of P and W. In Fig. 79, for example, a part of P (component p_0, Fig. 80) pushes the lever in the direction BF. The pressure on F in this instance would be the resultant of $p_1 + W$ acting vertically and p_0 acting horizontally.

* It was Leonardo da Vinci (1500) who added to the general principle of moments this second method of finding the effective moment of a force.

From these considerations of oblique forces it follows that the **principle of moments**, deduced on page 122 for parallel forces, has a much more general application than was there assumed. If any number of forces in the same plane act upon a body to produce equilibrium, the sum of the moments of these forces about any point in that plane is equal to zero. The point with reference to which the moment is taken need not be within the body, and the forces need not be parallel, nor is it necessary in our diagrams to represent the form or outline of the body; the single line that we have used to connect the points of application of the forces is sufficient for all necessary computations.

In the lever shown in Fig. 82 the fulcrum is called upon to offer a resistance of 16 pounds, which is equivalent to a force of 16 pounds acting upward, as if a string tied at F were pulled upward by such a force. Now, if we take any point, as A, the moments of the three forces 6, 16, and 10 with reference to this point will be *

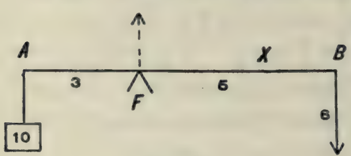


FIG. 82

$$6 \times 8 - 16 \times 3 \pm 10 \times 0 = 0.$$

With reference to F the sum of the moments will be

$$6 \times 5 - 10 \times 3 \pm 16 \times 0 = 0.$$

With reference to a point X two feet from B the sum of the moments will be

$$6 \times 2 + 16 \times 3 - 10 \times 6 = 0.$$

EXAMPLES

1. *Referring to Fig. 82, select any other point in the rod and show that with reference to this point the sum of the moments of the forces involved is zero.*

* A force tending to produce rotation in the direction of the hands of a clock is considered as having a positive moment; counter-clockwise a negative moment.

2. *Prove the same for a point at any given distance beyond either end of the rod.*

3. *Take a case of a couple of any given numerical value, and find the sum of the moments of the two forces with reference to several different points: say, one point in the line connecting the two forces; one in this line continued; and a third one at some convenient place entirely without this line. Is the sum of the moments the same in the three cases?*

Mechanical Advantage. — If a lever is used to enable a force of 6 pounds to balance a resistance of 10 pounds, there is an obvious advantage, but there is also the disadvantage that if motion takes place, the load will not be moved as far as the applied force. Conversely, if a machine is contrived so as to multiply the motion, as would be the case if a force were applied to the short end of a lever of the first order to move a load at the longer end, and always in levers of the third order, then the motion gained would be at the expense of force exerted, for the applied force would then have to be larger than the load in proportion as its leverage is less. We can choose either advantage, but we cannot accomplish both in the same machine; as already shown (p. 121) $Wh = Pk$, so that the applied force and the load cannot be otherwise than inversely proportional to the distances through which they would move.

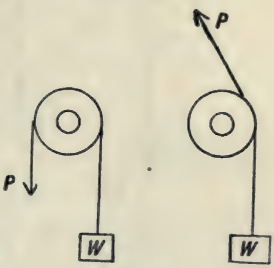


Fig. 83

Some mechanical devices, such as the lever with equal arms, may merely change the *direction* of the force or motion, without gain or loss of either. Likewise, the single fixed pulley (Fig. 83) permits no possible gain or loss of force, except by friction, but it can be used to accomplish any change of direction.

CHAPTER IX

MACHINES

The Wheel and Axle. — A pilot wheel will serve as a convenient type of the numerous devices in which the idea of the wheel and axle is employed. A comparatively small force applied to the handles on the periphery of the wheel is sufficient to overcome a greater resistance on the part of the tiller ropes.

A load W applied at the axle or drum represented by the small circle of radius r in Fig. 84 has a moment of $W \times r$

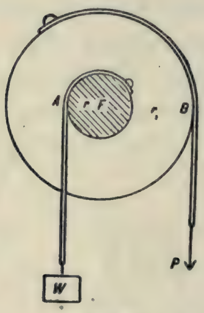


FIG. 84

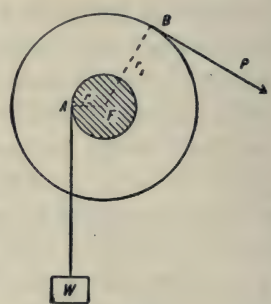


FIG. 85

with reference to the center F. If the "wheel" has a radius r_1, the applied force will be just sufficient to produce equilibrium if $P \times r_1 = W \times r$.

As a statical machine — one in which it is desired to maintain a condition of rest and equilibrium — the wheel and axle is no more useful than a simple lever. Even if P were applied at a point on the periphery of the wheel such that A, F, and B are not in the same straight line, as shown in Fig. 85,

the result would be the same if we had used a bent lever. The wheel and axle and the lever are thus associated with each other, because, for purposes of computation, the principle of moments is readily applicable to both. But as useful machines they present this difference, that the wheel and axle permits of continuous motion, while the available motion of the ends of a lever is practically limited to the small distance resulting from an angular change of less than

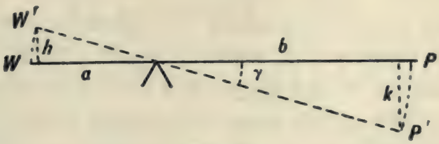


FIG. 86

90° around the fulcrum. When the lever has turned through an angle γ (Fig. 86), if the directions of P and W have remained unchanged, their moments will become $Pb \cos \gamma$ and $Wa \cos \gamma$, and are zero when $\gamma = 90°$.

When the wheel and axle is used, P and W do not follow the points B and A to a new position (Fig. 87) as rotation occurs, but are shifted to new points of application, in such manner that the angles WAF and PBF (Figs. 84, 85, and 87) may be kept equal to 90° — the position of maximum moment.

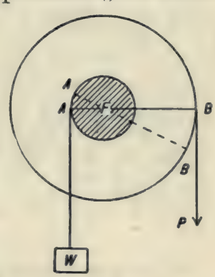


FIG. 87

It is not always convenient, however, to preserve this favorable direction of P and W with reference to their lever arms. In many contrivances, like the windlass, the wheel is replaced by a crank, which, acting like a single lever, loses the advantages of a wheel. The rope of a windlass is always at right angles to its instantaneous lever arm in the axle, but the force applied to the crank handle is not always exerted

to the greatest advantage, being sometimes at right angles to the crank and at other times acting at a less favorable angle. This is especially true of the crank of an engine, upon which the connecting rod acts at a varying angle. The spoke of a capstan is analogous to the crank of a windlass in construction, but a person operating a capstan, by walking in a circle, is always pushing at right angles to the lever arm, and hence nothing is lost by not having the entire wheel.

When a rope or belt acts on the circumference of a wheel an error is introduced in computing the moment unless we add to the radius of the wheel one half the thickness of the rope or belt; for, if the rope is wrapped around the circumference sufficiently to prevent slipping, it moves entirely with the wheel and the force is distributed over the entire cross section of the rope or belt. Perhaps this can be made a little clearer by the principle of work. As long as the rope is coiled around the wheel the inner strands are compressed, and the outer portions elongated, beyond their normal condition. If a force is exerted on the rope causing the wheel to turn, the distance through which the force moves is the length of rope unwound, after the compressed and elongated portions have returned to their normal condition. Hence, in determining the work done by the force in moving the wheel through any angle, we take as the distance through which the force moves an arc described with a radius equal to the radius of the wheel *plus* half the thickness of the rope.

EXERCISE

Give five examples of the wheel and axle, and if any involve the use of a crank, state whether the force acting on the crank maintains the position of maximum moment.

Gearing and Shafting. — The various combinations of shafts and pulleys, used so commonly in the transmission

of power, operate by a transmission of motion and force from one wheel and axle to another. The action of the entire system is traced from part to part in the same manner that a series of compound levers would be analyzed into its component simple levers.

For example, if a force of $P = 90$ pounds (Fig. 88) is applied at the circumference of a 24-inch pulley fastened on the same

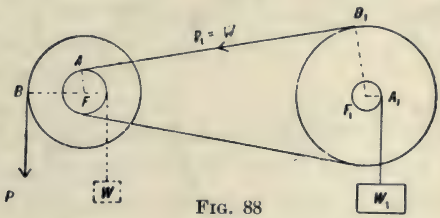


FIG. 88

shaft with a 9-inch pulley, it will be equivalent to an opposing force of $W = \frac{24}{9} \times 90$, or 240 pounds, applied to the smaller wheel. Now, instead of moving the weight W, suppose it is desired to convey this action to a second wheel and axle and at the same time to modify it farther. If it suits our needs, we may use a 30-inch pulley for the larger of these wheels and a 6-inch pulley for the smaller. By connecting the 9-inch pulley of the first wheel and axle with the 30-inch pulley of the second by means of a belt, these two wheels may be constrained to move with the *same lineal velocity*, and the force of 240 pounds at the perimeter of the 9-inch wheel is transferred through the belt to the perimeter of the 30-inch wheel. This force at B_1 will balance a load of $W_1 = \frac{30}{6} \times 240$, or 1200 pounds at A_1.

Exercise. — *Verify this result by the principle of work.*

When the speed is geared down to such an extent that very large forces are involved, it may not be practicable to use leather belting in the ordinary manner, because the belt

would slip on the pulley before such forces could be exerted If a rope is used it can be coiled around each pulley as many times as may be necessary to make it hold without slipping (provided, of course, that the rope is stout enough to stand the tension). Or, if the shafts can be placed close to each other the power can be transmitted from one to the other by means of cog wheels, as shown in Fig. 89, or by means of bevel gears if the shafts are not parallel. Since the number of teeth is proportional to the circumference of the wheel, the computed relation between P and W_1 will be the same as if we had used a belt, but the possibility of slipping is obviated.

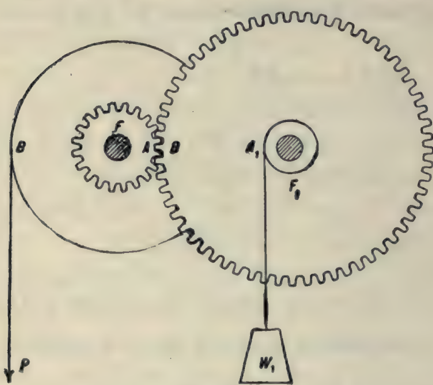


FIG. 89

Notice, however, that the direction of rotation of the second shaft is opposite to what it would have been if the transmission had been by belt.

From this figure the analogy between a system of gearing and a system of compound levers is readily observed. If BFA were a lever, instead of mere points in a wheel and axle, the downward force at B would cause an upward motion at A. From A the action would be transmitted to B_1, and the upward force on B_1 would have a lever arm B_1F_1, etc.

This holds true whether the transmission is by belting or by spur gearing. The only probable source of error in either case would be in misunderstanding the real relation between the 30-inch and 9-inch wheels; although these pulleys may be of different diameters, they do not give rise to the same mechanical considerations as if they were on the same shaft. Two pulleys of different diameters fixed on the

same shaft constitute a wheel and axle, in which case they would have the same *angular* velocities, while points on the respective perimeters would have different lineal velocities. But being mounted on different shafts, and one taking motion from the other at their perimeters, as in the instance under consideration, their perimeters have the same *lineal* velocity but different rates of rotation. Therefore, in accordance with the principle of work, the action is propagated from the 9-inch wheel to the 30-inch wheel without gain or loss of force.

The Pulley. — In machine-shop practice the word "pulley" is employed to designate a wheel over which a belt runs in a system of shafting, as explained under the wheel and axle. Long before the transmission of power by belts and ropes, the wheel was used in a somewhat different manner as one of the simple mechanical powers, and it is in this sense that we apply the generic term, "the pulley." Sometimes the advantage of the pulley consists only in a change of direction — a force applied in some convenient direction being employed to overcome an equal force acting in any other direction; under other circumstances, by a different arrangement of conditions, it may change the magnitude of the force or motion. Or, it may afford both of these advantages simultaneously, especially when several wheels are compounded in the same machine.

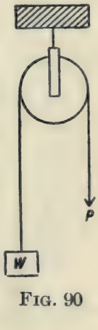


FIG. 90

Fixed Pulley. — Very frequently a single pulley, arranged in the manner shown in Fig. 90, is used for purposes of hoisting. A grooved wheel is pivoted in a framework, which is suspended from a rigid support. In the matter of equilibrium between P and W, friction in this instance is an important consideration. But disregarding this, it is obvious that according to the principle of virtual work $P = W$.

This is still true if the applied force is exerted in any direction, other than vertical; or even if the free end of the rope is carried by means of other *fixed* pulleys to any distance where it may be convenient to apply the action.

EXERCISE

In the case of the fixed pulley referred to in Fig. 90, in which both P and W act vertically downward, what total load is the rigid support required to sustain?

Movable Pulley. — If a single pulley is arranged in such manner that one end of the rope is fixed to the support, while the pulley itself is free to move with the weight, the relation between P and W is greatly changed. Also, in this case allowance has to be made for the weight of the pulley as well as for friction. But assuming a weightless pulley and neglecting friction, P will be equal to $\frac{W}{2}$, because if motion takes place the free end of the rope will move twice as fast as the weight. To make this clear, imagine that the entire pulley and weight were grasped in the hand and raised to a height of one foot. The rope would then be left dangling for a length of one foot on each side, and to take up the slack, the free end, to which P is applied, would have to be raised two feet.

FIG. 91

Looked at as a simple question of statics, it will be observed that the total weight is supported equally by the two ends of the rope.

If the two parts of the rope are not parallel, the relation $P = \frac{W}{2}$, just deduced for the single movable pulley, is no longer true. If P is applied in a direction that is not parallel with the direction of W, the pulley and weight will

not remain vertically below the point of support, but will roll to one side until they reach a position where the two parts of the rope make equal angles with a vertical line through the center of gravity of the pulley and weight (Fig. 92). The two parts of the rope will still be under equal tensions, and the load P_b at the point of support, in the direction of the rope on that side, will still be equal to the applied

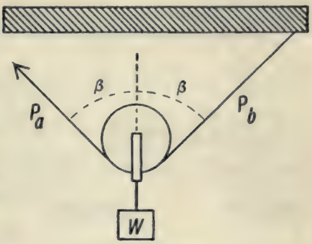


FIG. 92

force P, but P will not be equal to $\frac{W}{2}$. We must treat P_a and P_b as two component forces, and to produce equilibrium, their resultant must be equal and opposite to the known force W. Hence, knowing the values of W and β, and knowing that

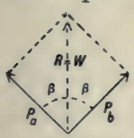


FIG. 93

the two components P_a and P_b are equal, we can compute the value of the applied force from the relations of the parallelogram (Fig. 93). If R represents this resultant, equal and opposite to W, then

$$R = 2\,P_a \cos\beta\,;$$

whence
$$P_a = \frac{W}{2\cos\beta}.$$

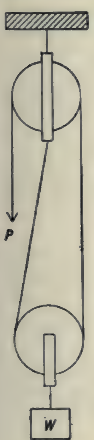


FIG. 94

FIG. 95

Combinations of Pulleys. — There are two ways, illustrated in Figs. 94 and 95, by which a fixed pulley and a movable one may be used in combination. Later on it will be shown that these combinations are not essentially different, but for the present we will consider them separately. In the first case only one end of the rope moves when the weight is raised, the other end being attached to the fixed pulley. In the other

case every part of the rope moves when the machine is in operation.

In the first case the applied force would move twice as fast as the weight, and hence $P = \frac{W}{2}$. For "gaining power" this arrangement of the two pulleys has no advantage over the single movable pulley, but it has the available advantage that by passing the rope over the fixed pulley, the force is exerted downward instead of upward. This difference is shown in Figs. 96 and 97, placed side by side. If the rope were not passed over the upper pulley, as shown by the dotted line, this pulley would serve no purpose whatever; and even when it is called into use the free end of the rope does not move any faster for that reason, and there is no further gain of power,— which is true, as we have learned, of *all* fixed pulleys. The weight drags down on two ropes, each of which supports $\frac{W}{2}$, and the tension in the free end of the rope merely balances one of these.

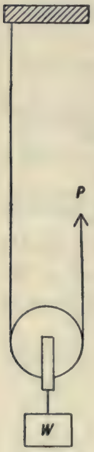


FIG. 96

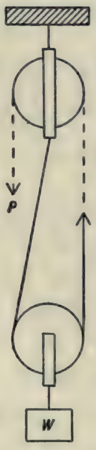


FIG. 97

In the second case (Fig. 95), where one end of the rope is attached to the movable pulley, $P = \frac{W}{3}$, as may be readily proved by the principle of work. Statically, this follows from the fact that the weight is supported equally by the three ropes.

In these two cases the load sustained by the point of support is very different. In the first case it is $\frac{3W}{2}$; in the second case it is $\frac{2W}{3}$. Because, in the first case there are three

ropes dragging downward and each is under a tension of $\frac{W}{2}$; while in the second case there are only two ropes, whose tensions, each $\frac{W}{3}$, exert a downward force upon the support. In each system if we replace the weight by a rigid support, the pull will be $3\ P$ on one support and $2\ P$ on the other. In other words, the second system is the same as the first, reversed end for end.

From the relations just deduced it will be observed, as a general principle of pulleys, that if we disregard friction, etc., the tension is the same in all parts of the rope; * and this tension is equal to the quotient of total load at either end of the system — either the weight, or the load sustained at the point of support — divided by the number of parts of the rope at that end. For example, in the first of the two cases just considered, the load on the support was $\frac{3\,W}{2}$, in the same ratio as the number of ropes at the support and at W, $3:2$; and the tension throughout the rope is $W \div 2$, or $\frac{3\,W}{2} \div 3$, the former being the quotient for one end of the system, and the latter being the quotient for the other end. In the second case the load on the supporting point was $\frac{2\ W}{3}$, the number of ropes at the two ends of the system being in the same ratio $2:3$, and the tension being $\frac{W}{3}$.

From this consideration it follows, as we have already asserted, that there is no essential difference between these two cases; when a pull is exerted on the free end of the rope, a force is called into action at each end of the system of pulleys. We call the "movable pulley" the one that moves first. The fact that the other end did not move, or

* This does not apply to the *different* ropes used in the systems illustrated in Figs. 99, 100, and 101.

that the two ends did not move together, is a circumstance entirely independent of any consideration essential to the pulley. Since pulleys are used to overcome some resistance by fastening one end of the system to a fixed support, the other or movable end is always the one at which the "weight" W is assumed to be placed.

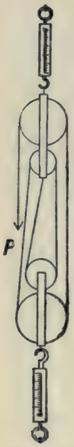


Fig. 98

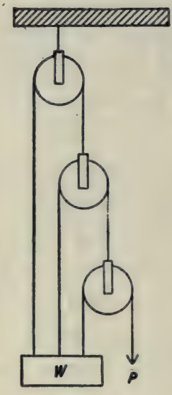


Fig. 99

EXAMPLES

1. *A system of four pulleys is arranged in the manner shown in Fig. 98, and the two ends are attached to spring balances. If a pull of 2 pounds is exerted at the free end of the rope, what will be the reading of each of the spring balances?*

Disregard friction, weight of pulleys, etc. In practice, the pulleys would be of the same size and would be placed side by side in each "block." They are shown differently in the diagram for the sake of clearness.

2. *When the upper balance reads 100, what is the magnitude of the applied force, and what is the reading of the other balance?*

3. *Find the relation between P and W, the tension in each rope, and the load on supporting beam, for the system of pulleys shown in Fig. 99.*

4. *Find the relation of P and W, the tension in each rope, etc., for the system shown in Fig. 100.*

5. *Find the relation between P and W, tensions in ropes, etc., for the system shown in Fig. 101.*

The Inclined Plane. — As applied to tools and machines, the inclined plane is represented in chisels and other edge tools, nails, screws, wedges, cams, eccentrics, propeller blades,

etc. The name is derived from the primitive device of a plane surface used to elevate a "dead load" to some desired altitude by means of any "easy" incline — a skid, for example.

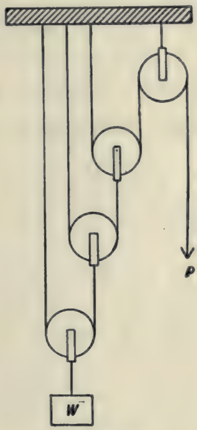


Fig. 100

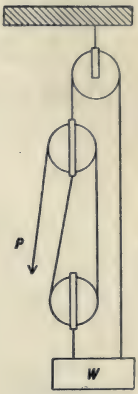


Fig. 101

The steepness of a gradient or slope is sometimes designated by the number of degrees in the angle between the inclined plane and the horizon, but it is more commonly expressed as a percentage. This percentage is the quotient of $\frac{BC}{AB}$ (Fig. 102), the altitude and base of a right triangle, of which the hypotenuse is any part of the inclined surface, and equals tan β.

The mechanical advantage of the inclined plane depends upon the steepness of the gradient and the direction in which the applied force is impressed. Let the right triangle ABC (Fig. 103) represent an inclined plane, of which AB is the horizontal base. Let the base, altitude, and hypotenuse be

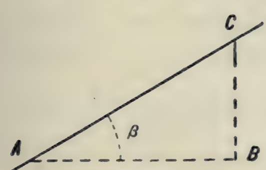


Fig. 102

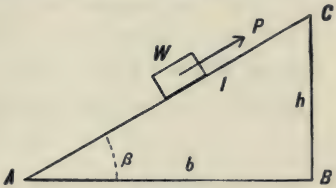


Fig. 103

represented by b, h, and l respectively. To prevent the weight W from sliding down the plane let a **force be applied in a direction parallel to the slope,** or hypotenuse, AC. If there were no friction, what would be the relation between P

and W to produce equilibrium? This problem can be solved by either of two methods: (1) by the principle of work, or (2) by resolving the weight W into two components parallel and perpendicular to the inclined plane.

1. By the principle of work.

Since gravity acts vertically, no work is done in moving a body horizontally with a uniform velocity (except against friction). Hence the work done *against* gravity in moving W from A to C is Wh — the same as if it had been raised vertically from B to C. The work done *by* P is measured by the distance moved in the direction in which it acts, and is equal to Pl.

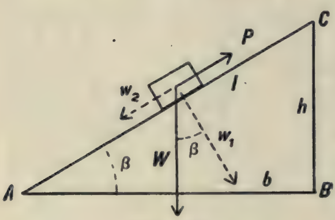


FIG. 104

By the principle of work $Pl = Wh$,

or
$$P = \frac{h}{l} W.$$

2. By resolving the vertical force W.

If we resolve W into two components, w_1 and w_2 (Fig. 104), one perpendicular and the other parallel to AC, the former will have no tendency to move the weight one way or the other along the plane. The latter component w_2 is the only part of W that P is required to equilibrate.

Hence, to produce equilibrium

$$P = w_2 = W \sin \beta.$$

From the triangle ABC,

$$\sin \beta = \frac{h}{l}, \text{ whence}$$

$$P = \frac{h}{l} W, \text{ as already proved.}$$

When P is not parallel to the Plane. — When P is applied in a direction making an angle γ with the hypotenuse, as shown in Figs. 105 and 106, it is only partially effective in the direction AC, the component p_1 perpendicular to AC being entirely useless in any direction at right angles to itself (except to alter the amount of friction between W and the plane, which we are now disregarding). If p_2 is the component of P parallel to AC, then by the last paragraph

$$p_2 = \frac{h}{l} W = W \sin \beta.$$

But $p_2 = P \cos \gamma$, whence $P \cos \gamma = W \sin \beta$, or $P = W \dfrac{\sin \beta}{\cos \gamma}$.

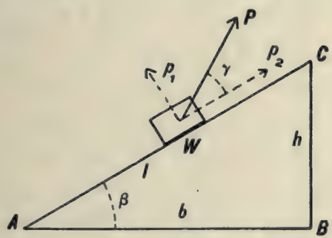


Fig. 105

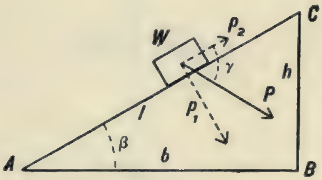


Fig. 106

A very important case arises when the direction of P is parallel to AB. — The angle γ is then equal to β, and if $\beta = \gamma$ the expression

$$P = W \frac{\sin \beta}{\cos \gamma},$$

becomes
$$P = W \frac{\sin \beta}{\cos \beta} = W \tan \beta.$$

Referring to the triangle it will be seen that

$$\tan \beta = \frac{h}{b}, \text{ whence } P = \frac{h}{b} W.$$

That is, if the power is applied parallel to the base, P and W are in the inverse ratio of the base and height. This, of course, is verified by the principle of work; for when the weight is moved up the plane from A to C, it moves through

the horizontal distance b and the vertical distance h; and P being applied horizontally does an amount of work equal to Pb, which we know must be equal to Wh. If $Pb = Wh$, then

$$P = \frac{h}{b} W.$$

The Wedge. — It has just been shown that when the "weight" on an inclined plane acts in a direction perpendicular to the base, while the power is applied parallel to the base, the relation between P and W is $P = \frac{h}{b} W$. A practical illustration of this would be the device shown in Fig. 107. A rod fitted into the guides G and G_1, and hence free to move only in a vertical direction, is held down against an inclined plane by means of a weight W. A pressure P is applied to the back of the inclined plane, which is pushed, wedgelike, under the rod. If the first position is such that point A is vertically under the rod, and the plane is pushed through its entire length, until C comes under the rod, the latter will have been raised through a vertical distance BC, while P moves through the horizontal distance BA. By the principle of work $Pb = Wh$, or $P = \frac{h}{b} W.$

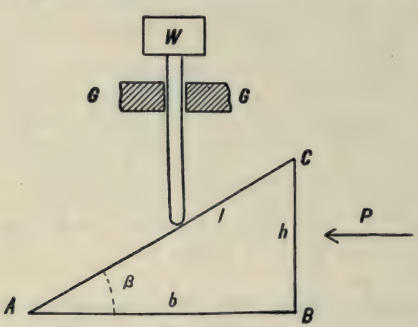


FIG. 107

It happens, however, in devices where the wedge is used, that W is not always, or even usually, vertical, — that is, perpendicular to the base of the plane. Very frequently it is perpendicular to the hypotenuse, as in the case of a carpenter's chisel used to split a piece of wood in the

manner shown in Fig. 108. If the angle at the edge of the chisel is β, and the cohesion of the wood causes a pressure W at right angles to the slope, then the conditions of the problem are as represented in Fig. 109. Having resolved W into components perpendicular and parallel to AB, the problem can be solved by statics or by the principle of work.

Statically, the component w_2 is the only part of W opposing P and tending to prevent the wedge of the chisel from

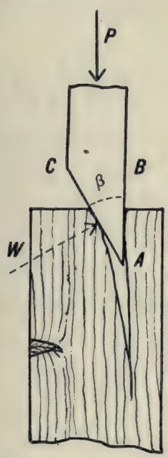


FIG. 108

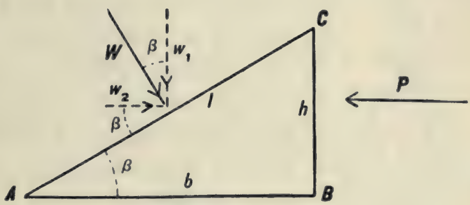


FIG. 109

pushing into the wood (for the component w_1 can exert no influence in the direction of P, at right angles to itself). To produce equilibrium $P = w_2$, and since $w_2 = W \sin \beta$ then $P = W \sin \beta$.

Or, *by the principle of work*, if the wedge of the chisel is pushed into the wood from A to C, the spreading of the wood through a distance BC will signify that the component w_1 is overcome through that distance, requiring an amount of work, or expenditure of energy, equal to $w_1 \times h$. (The component w_2 is not now considered, for the reason

that its point of application is not moved through any distance in its own direction — that is, parallel to AB.) Since the force P, parallel to the base, does this work by moving through the distance BA, or b, we have $Pb = w_1 h$,

or
$$P = \frac{h}{b} w_1.$$

But
$$\frac{h}{b} = \tan \beta, \quad \text{and } w_1 = W \cos \beta.$$

Whence by substitution, $P = W \sin \beta$, as already proved by the static method.

In most cutting tools the edge has the V-form of the typical wedge, illustrated in Fig. 110. A wedge of this sort is obviously equivalent to two inclined planes having a common base, as represented by the vertical dotted line. By driving the wedge full length into the wood, the fibers are spread twice as far as if the wedge had been a right triangle instead of isosceles. The resistance W is taken as being perpendicular to each face. Therefore, the relation between P and W is $P = 2\, W \sin \beta$. The same relation is sometimes expressed in terms of the total angle at the vertex of the wedge, or $P = 2\, W \sin \frac{\gamma}{2}$.

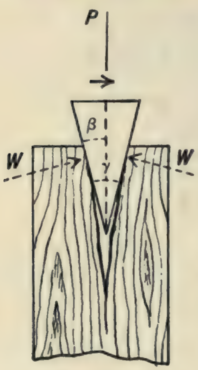


FIG. 110

In the ordinary use of a wedge for rough work — such, for instance, as we have selected for our illustration — the amount of friction is very great. But in cases where accurate calculations would be at all necessary the friction would be reduced to a reasonable limit, where it could be taken into consideration and properly allowed for in the computations, as in the "V" ways of a lathe or planer bed.

The results of the preceding computations for the inclined plane and wedge are summarized in the following table, showing the

relations between P and W for the different directions in which these forces usually act:

$$1.\quad \left.\begin{cases} W \text{ perpendicular to base} \\ P \text{ parallel to hypotenuse} \end{cases}\right\} P = \frac{h}{l} W = W \sin \beta.$$

$$2.\quad \left.\begin{cases} W \text{ perpendicular to base} \\ P \text{ parallel to base} \end{cases}\right\} P = \frac{h}{b} W = W \tan \beta.$$

$$3.\quad \left.\begin{cases} W \text{ perpendicular to hypotenuse} \\ P \text{ parallel to base} \end{cases}\right\} P = \frac{h}{l} W = W \sin \beta.$$

$$4.\quad \left.\begin{cases} \text{For ``V'' wedge} \\ W \text{ perpendicular to sides} \\ P \text{ perpendicular to back} \end{cases}\right\} P = 2\frac{h}{l} W = 2\, W \sin \beta.$$

The Screw. — The screw is a combination of the inclined plane (in a modified form) and the wheel and axle. The wheel-and-axle element is illustrated by taking a forged bolt before the thread is cut and applying a wrench to the head. The wrench will turn in a large circle, constituting the "wheel," and the shank of the bolt will be the "axle." Cutting the thread adds the element of the inclined plane, as may be shown by the following simple experiment. Cut a

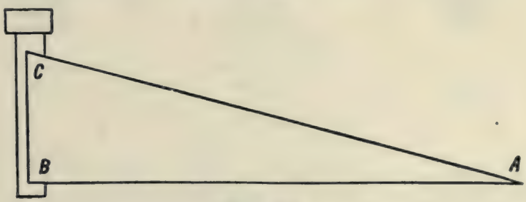


Fig. 111

piece of paper into triangular form to illustrate an inclined plane. The grade or slope of this plane will depend upon the diameter of the bolt and the desired pitch of the screw. Place the edge BC of the plane against the side of the bolt and parallel to the axis of the cylinder. By wrapping the paper continuously around the bolt, the upper edge CA, representing the inclined surface, will describe a helix, which

will coincide with the path along which a thread could be cut of the desired pitch.

The pitch of a screw could be defined in the same manner as an ordinary inclined plane, — as an angle measured in degrees, or as a percentage, — but it is easier to make computations directly from the number of threads per inch, measured parallel to the axis. A machine screw is designated by two numbers (besides the length), one referring to the diameter of the shank and the other giving the number of threads per inch. A number 16 pitch means 16 threads per inch.

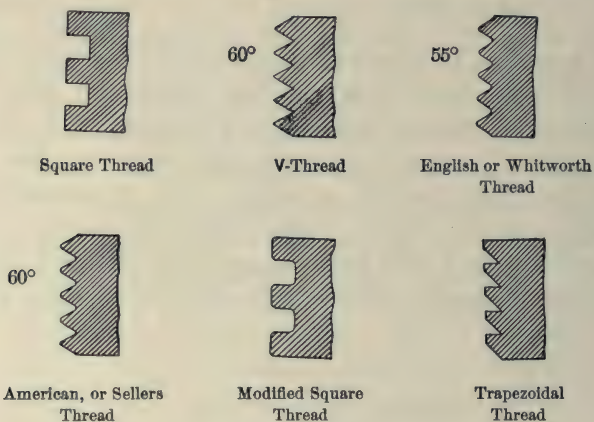


FIG. 112

The form of a thread is sometimes rectangular, sometimes V-shaped, and sometimes a modified form with rounded edge or vertex. But the shape or angle of the thread does not alter the pitch.

The operation of a machine screw or a bolt requires a **nut**. Both bolt and nut may be movable, or one of them may be fixed, according to needs. A wood screw is always used in soft material, which is compressed into necessary grooves by the thread of the advancing screw.

When a force is properly applied to the screw, the action

of the thread is analogous to that case of the inclined plane where P is applied to the back of the plane and W acts perpendicularly to the base (Fig. 107, p. 150). There is the additional consideration, of course, of the wheel and axle. It is not necessary, however, to follow out all these relations, because the total mechanical advantage of a screw can be computed directly from its pitch by the principle of work. Suppose, for instance, that a bolt of $\frac{1}{2}$ inch diameter, and number 8 pitch, is turned by means of a wrench applied to the head. The resistance W, let us assume, is occasioned by compressing two pieces of wood held together between the nut and the head of the bolt. If a pressure P is exerted on the wrench at a point 9 inches from the axis of the bolt, what total resistance W can it overcome? One complete turn of the wrench will cause the bolt to advance through the nut $\frac{1}{8}$ inch (the distance between two adjacent threads), or $\frac{1}{96}$ foot. The distance moved by the point at which P is applied will be

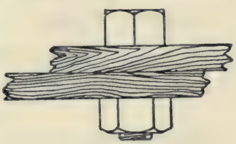


FIG. 113

$$\frac{2\pi \times 9}{12} \text{ feet.}$$

Therefore $$P \times \frac{2\pi \times 9}{12} = W \times \tfrac{1}{8} \times \tfrac{1}{12}.$$

In general, if h is the distance between two adjacent threads, and r is the distance from the axis of the screw to the point of application of P, the relation between P and W is expressed by the formula

$$P \times 2\pi r = Wh.$$

The two distances r and h must be expressed in the same units, — both in inches or both in feet. In the problem taken as an illustration these distances were reduced to feet in order to get the foot-pound as a unit of work.

Notice that W is supposed to act parallel to the axis of the screw, which would be perpendicular to the base of the inclined plane.

Notice, also, that the diameter of the bolt is not used in the calculations. It is duly involved, however, in the pitch, but is eliminated from the computations when the mechanical advantage is deduced directly from the pitch. The pitch h is equal to $\pi d \tan \beta$, where d is the diameter of the bolt and β is the slope of the thread measured in degrees. Each thread is equivalent to an inclined plane of height, h; base, πd; and slope β, wrapped around the bolt.

Endless Screw. — Consider a threaded cylinder fitted into guides in the manner shown in Fig. 114. The screw itself cannot advance, being restrained by the guides, but the toothed wheel D can be made to rotate continuously by turning the screw. The teeth on the circumference of the wheel feed into the threads of the screw from one side and out at the other. From the wheel D power may be taken, after the manner of the wheel and axle, and the action is endless, notwithstanding the limited length of the screw. On account of the great reduction of velocity obtained by this combination its mechanical advantage is very great. It is much used in hoisting machinery. It is also useful in making fine angular adjustments, as in gear-cutting machines; when thus used, it is called a tangent screw. It is also called the worm and wheel.

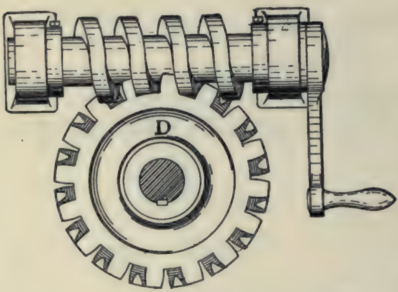


FIG. 114

The Cam and Eccentric. — If a circular disk is pivoted at a point other than the center, as the point O (Fig. 115), and

is made to rotate around that point, it is said to have an eccentric motion, and is itself called an eccentric. If a rod were set in guides so as to rest vertically upon point S, it would be pushed up vertically from S to T' by a semi-rotation of the eccentric. If the eccentric is made to rotate continuously, the rod will move up and down with a reciprocating motion between positions S and T'.*

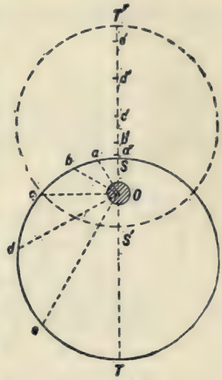


FIG. 115

The total work done in raising the rod from S to T' is independent of the manner in which it is accomplished, but it is not proportioned uniformly throughout the stroke. It is equivalent to pushing a weight up an inclined surface of varying slope, or rather pushing such a surface under the weight. Suppose, for instance, that the lines Oa, Ob, Oc, etc., form six equal angles around O. As rotation takes place, the points a, b, c, d, e, and T assume positions a', b', c', etc., successively. During the first $\frac{1}{12}$ rotation the rod is raised through the distance Sa', equal to the difference between Oa and OS; during the next interval of $\frac{1}{12}$ rotation the rod is raised from a' to b', the difference between Ob and Oa, etc. Since $a'b'$ is greater than Sa', it is obvious that the average slope of the surface pushed under the rod during the second interval was greater than during the first interval.

The distance ST' is called the "swing" or "throw" of the eccentric.

A **cam** is essentially the same as an eccentric, differing

* The eccentric is used upon engines to produce and control the valve motion, but the motion thus given to the eccentric rod is not the same as that given to the rod mentioned in the above illustration. A valve eccentric, running in an eccentric strap, gives a motion identical with that of a crank of the same swing.

mainly in the shape of the periphery. By varying the surface of the disk an infinite number of straight-line motions can be produced. By means of a certain heart-shaped disk


FIG. 116


FIG. 117

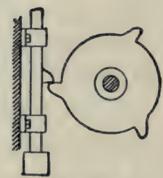


FIG. 118

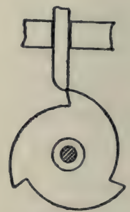


FIG. 119

(Fig. 116) the rod can be given a uniform velocity. If the disk is not symmetrical, the return stroke will be different from the out stroke (Fig. 117). Several repetitions of the straight-line motion can be accomplished in a single rotation of the axis, by means of lugs or shoulders on the edge of the

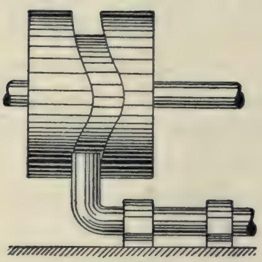


FIG. 120

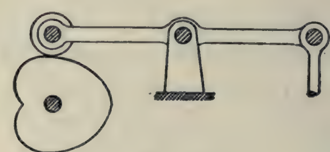


FIG. 121

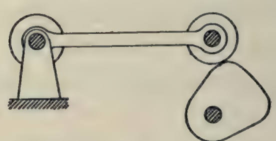


FIG. 122

disk (Figs. 118 and 119). Sometimes the end of the rod plays in an irregularly grooved collar (Fig. 120). All these cases involve the inclined plane and can be solved by the principle of work if the swing is known.

The end of the rod that bears on the cam is usually fitted with a roller, for which allowance must be made in constructing the cam to give a required motion to the rod. Very frequently the rod is connected with a system of levers to still further modify the motion (Fig. 121). By the arrangement shown in Fig. 122, uniform motion of the shaft bearing the eccentric may be converted into irregular motion of a second shaft.

The Toggle Joint. — The toggle joint is used in adjustable carriage tops, in printing presses, in machines used for stamping metals, leather, wood, etc., and occasionally in copying presses when the desired pressure is greater than

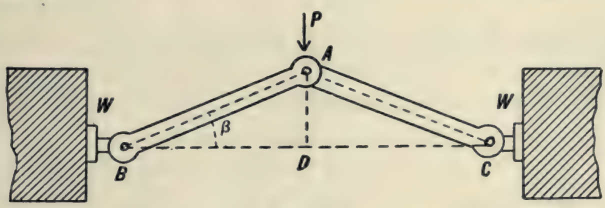


FIG. 123

could be readily obtained from a screw. Its mechanical advantage is enormous.

Two bars AB and AC are pivoted in the manner shown in Fig. 123, and the ends B and C are constrained to move laterally, or at right angles to the path of A. A small pressure at A, applied as shown, will produce a very great outward pressure at B and C, depending upon the magnitude of the angle β.

The relation between P and W is not fixed, but changes as the angle of the joint changes, the mechanical advantage of P over W becoming greater and greater as the rods flatten out. If a constant force P is applied at A, it will give a certain pressure W at B and C, but as the joint flattens out, gradually approaching the straight line BC, the value of W becomes greater and greater, though the applied force P has

remained constant. As point A moves through the distance AD, the end B moves through the difference between AB and DB. Now $DB = AB \cos\beta$, and for small angles $\cos\beta$ is almost at its maximum value, — that is, very nearly equal to unity, — whence the difference between AB and AD is very small. Even when β is as large as 5°, we find $\cos\beta = 0.9962$, and $AB - BD = 0.0038AB$. As BAC approaches a horizontal position as a limit, this difference becomes almost infinitesimal in comparison with an appreciable movement of P in direction AD, and therefore, at this instant — just before β becomes zero, according to the principle of work, W is enormously greater than P.

Differential Motion. — In the differential pulley, the differential screw, and the differential wheel and axle, an extra part is added to the machine with the sole function of reducing the motion of the load and thereby increasing the mechanical advantage of the machine.

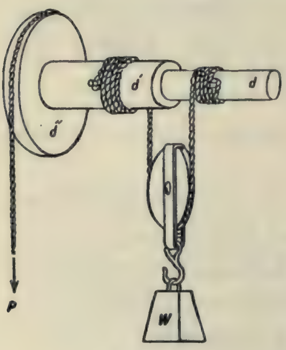


FIG. 124

In Fig. 124, illustrating a differential wheel and axle, P is applied to the circumference of a wheel of diameter d''. The weight is suspended from a pulley which is itself supported by ropes coiled in opposite directions around the cylinders d and d', constituting two portions of a solid or continuous axis, upon which the wheel d'' is fixed. If the force P causes the wheel and axle to rotate, the rope wrapped around d' will be drawn up, but at the same time the portion coiled upon the cylinder d will be unwound. The rate at which W will be raised will depend upon the difference between the circumferences of the two parts of the axis (though we must remember also that the pulley supporting

the weight is a movable pulley, for which reason the weight rises only half as fast as the rope is shortened).

Everything duly considered, it will be found that

$$P \times \pi d'' = W \times \left(\frac{\pi d' - \pi d}{2}\right), \text{ or}$$

$$P \times d'' = W\left(\frac{d' - d}{2}\right);$$

$$P = W\frac{d' - d}{2d''}.$$

The chain hoist (or differential pulley, as it is called) and the differential screw are perhaps the most familiar devices in which the idea of differential motion is used. In the differential pulley, two wheels of slightly different diameters are fixed on the same axis, after the manner of a wheel and axle. A third wheel supports the weight in the manner shown in Fig. 125. Instead of a rope, as ordinarily used with pulleys, an endless chain passes from pulley d' to the movable pulley (part h); thence to the pulley d (part k); thence downward, hanging free (part l); and thence to the circumference of d' (part m), thus completing the circuit.

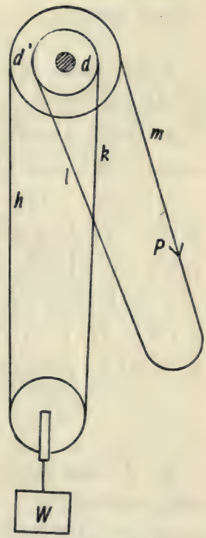


Fig. 125

Now if P is applied to part m of the chain, the fixed pulleys d and d' will rotate together; part h of the chain will move upward as far as P moves downward, and at the same time part k is fed downward from the circumference of d. But since d and d' are fixed to each other and rotate together, the part h is taken up faster than part k is fed downward, and hence in a single rotation of these two wheels W will be raised a distance equal to

one half the difference between their circumferences, and P will move downward through a distance equal to the circumference of the larger. The conditions are the same as in the case of the differential wheel and axle, except that P acts from the circumference of d' and not from a separate wheel. That is $d'' = d'$, whence

$$P = W\frac{d' - d}{2\,d'}.$$

The difference in size of the two wheels or cylinders that contribute to the differential motion can be made so slight that the motion of W may be made as slow as we please, and thus a very small applied force can be made to lift an enormous weight, up to the limit of the strength of the machine.

Compound Machines. — Many hand tools involve only one of the different mechanical powers enumerated on page 119, but most mechanical contrivances are more complex. A compound machine is one made up of a number of simple machines. The "weight" of the first becomes the "applied force" of the next, as in compound levers (p. 126), and the train of wheels (p. 140), though it is not necessary that all the different parts of the machines should be of the same kind. In moving a house on rollers, for example, a capstan is used in combination with a compound system of pulleys.

EXAMPLES

1. *If two forces of 60 pounds each, acting at an angle of 60° with each other, are exactly balanced by a third force, making an angle of 150° with each of the two given forces, find the magnitude of the third force.*

2. *A straight uniform lever AB, 12 feet long, balances at a point 5 feet from B, when weights of 9 pounds and 13 pounds are suspended at A and B respectively. Find the weight of the lever.*

3. *An iron bar 6 feet long is free to turn about a horizontal axis which is 4 feet from one end. On top of it is placed a second bar*

4 feet long, but otherwise like the first bar, so that the lengths of the bars are parallel. The two bars are now balanced upon the pivot. Describe the position of the top bar.

4. *Analyze the action of a claw hammer in drawing a nail. Where is the fulcrum? The "weight"? The "power"?*

5. *Why does the driver of a heavy load take a zigzag course in climbing a hill?*

6. *The drum of a windlass has a diameter of 10 inches, and the crank has a radius of 18 inches.*

(*a*) *What minimum force must be applied to the handle to raise a load of 160 pounds?*

Disregard friction and thickness of rope.

(*b*) *What difference if we allow a half-inch diameter of rope?*

7. *The screw of a letter press has a pitch of $\frac{1}{4}$ inch, and the diameter of the wheel is 10 inches. What pressure will be exerted upon the copying book by two forces of 25 pounds each applied tangentially at the circumference of the wheel?*

8. A two-arm balance is supposed to have arms of equal length; otherwise it is false. A false balance may appear to be true because the beam comes to rest horizontally when there is no load in the pan, but a little thought will show that it will not continue at equilibrium when equal weights are added.

(*a*) *If a dishonest dealer were using a false balance of this sort, in which arm would he place the substance being sold?*

(*b*) *Can you think of any way by which to test a balance to determine whether it is true or false?*

(*c*) *If w_1 is the apparent weight of a given object in one arm, and w_2 is its apparent weight in the other arm, prove that its true weight is $w = \sqrt{w_1 w_2}$.*

9. *In what way do metal-cutting shears differ most from ordinary scissors? Why?*

10. *In the chain hoist described on page 161, the two wheels d and d′ have diameters of $6\frac{1}{2}$ inches and 7 inches respectively. What force P must be applied for a load W equal to 600 pounds?*

11. *A 1200-pound anchor is hoisted by means of a capstan having a drum of 18 inches diameter and four spokes each 8 feet long. What force must be exerted by each of four sailors pushing at the extreme ends of the spokes?*

12. *A plank lies with one end projecting over a log. A boy weighing 100 pounds walks out on the projecting end, and when he gets 6 feet from the log, the plank tips. The center of gravity of the plank is 5 feet from the log. Find the weight of the plank.*

13. *A plank weighing 200 pounds lies with one end projecting over a log. A boy weighing 100 pounds walks out on the projecting end, and when he gets 6 feet from the log the plank tips. Where is the center of gravity of the plank?*

14. *A house on rollers is moved by means of pulleys and a capstan.*

(*a*) *If the resistance to rolling is 20 tons, at what rate can it be moved by a single horse working at the rate of one horse power.*

(*b*) *If the drum of the capstan has a diameter of 20 inches and the system of pulleys has two wheels in each block, what force must the horse exert at a point on the arm of the capstan 14 feet from the center of the drum? At what rate must the horse walk in order to move the house at the rate computed in part* (*a*).

15. *For the endless screw shown on page 156, assume the following dimensions:*

Pitch of screw, 1 inch.
Crank arm, 12 inches.
Number of teeth in wheel D, 16.
Diameter of shaft bearing wheel, $1\frac{1}{2}$ inches.

What force must be applied to the crank handle to lift a load of 500 pounds applied to the perimeter of the shaft?

CHAPTER X

FRICTION

When one body moves in contact with another, whether by sliding or by rolling, or when an object travels through a fluid medium, as a bullet through the air, we instinctively assume that a frictional resistance always accompanies the motion. The laws of friction are simple in statement, but they are quite empirical and cannot always be applied with that rigid mathematical accuracy which characterizes other laws of mechanics. This is because the usual phenomenon of friction, simple as it seems, is really an aggregation of a great number of less obvious phenomena, or of several groups of such aggregations, which taken together give rise to a complexity of conditions surpassing the possibility of mathematical analysis. For example, when one surface slides over another the frictional resistance is supposed to be occasioned mainly by the interlocking of an infinite number of particles on the two adjacent surfaces. No surfaces are smooth enough to be entirely without friction; two "perfectly smooth surfaces" (if such were possible) when placed in contact with each other, and the air excluded, would be close enough together for complete intermolecular action between the two surfaces, and the two bodies would cohere if of the same material (or adhere if of different materials) so firmly that they would be as a single body. No doubt the friction between two ordinarily smooth surfaces is caused in part by molecular attractions as well as by the interlocking of physical particles, or particles having appreciable dimensions.

Still more complicated is the frictional resistance between

lubricated surfaces, or of a moving carriage. The rolling friction of the tires in contact with the ground, the sliding friction in the bearings, and the frictional resistance of the air (each in itself a complex phenomenon) are all included in the total tractional resistance of the carriage.

It will be convenient to consider in order the laws (*a*) of sliding friction; (*b*) of rolling friction; (*c*) of the friction of ropes and belts on pulleys; and (*d*) the use of lubricants.

Sliding Friction. — For the purpose of illustrating the laws of sliding friction, the following results, obtained by actual experiment, may be used :

A planed cedar board was placed horizontally, and on it was placed an ordinary building brick, ground smooth on the face in contact with the board. A string passed around the brick was attached to a spring balance. By pulling horizontally on the balance it gradually reached a tension at which the brick began to slide. Repeating the experiment, it was found to move again at about the same reading of tension on the balance, — 2.5 pounds.

A block of Oregon pine, of quite different dimensions from the brick, was placed on the cedar board as nearly as possible under the same conditions as the brick, and was found to slide when the pull was only 1.1 pounds.

How shall we account for this difference — 2.5 pounds to slide the brick and only 1.1 pounds to slide the pine block? Was it due to the difference in the weights of the two; or to a difference in the areas of the surfaces of contact; or to a difference in the natures of their surfaces; or to all these causes in varying degrees?

To answer these questions the observations were continued as follows:

(1) A second pine block of exactly the same dimensions as the brick was loaded with weights until it had also the same weight as the brick, 5.6 pounds. Placed on the cedar

board under the same conditions as before, it was found to slide under a pull of 3.8 pounds.

Now, since the brick and the block have the same weight and the same area of surface in contact with the board, the difference between the friction of the brick and cedar (2.5 pounds) and the friction of the pine block and cedar (3.8 pounds) must be due to the fact that the brick differs from the pine in the nature and condition of its surface.

In general, the friction tending to prevent sliding between two surfaces is different for different pairs of substances. It has been found in practice that **friction is generally greater between surfaces of the same kind**, as between steel and steel; leather and leather; etc. Hence the advantage of using brass bearings for steel shafts, and of covering with leather the face of a pulley used with leather belting.

(2) The opposite side of the brick had not been ground. Placing the brick on the cedar board with the rough side downward, the friction was found to be 2.8 pounds, as against 2.5 for the smoothed surface.

The sliding **friction depends upon the roughness of either or both of the surfaces** in contact. It is obvious, however, that this roughness cannot be measured mathematically.

(3) One of the long, narrow faces of the brick had been ground and the opposite face left unground. Placed with the smooth edge on the cedar board the friction was found to be about 2.5 pounds, the same as for the broad side similarly ground. The friction for the unground edge was the same as for the rough side of greater area.

In general, the **friction between two surfaces is independent of the area of contact.** This law is true within broad limits; but when one of the bearing surfaces is so small that it digs or cuts into the other, or when the pressure is so great that the surfaces are deformed or abraded, the law is interfered with by new considerations, quite distinct from frictional influences.

(4) The most important law — that the **friction between two surfaces is proportional to the force pressing them together** — is deduced from the next observations.

The Oregon pine block used above was loaded up to a total of 4 pounds, including the weight of the block. The friction determined in the same manner as before was found to be 2.6 pounds. By placing additional weights on the block, up to a total of 8 pounds, the friction was increased to 5.25 pounds, or about twice as much as before. In the same manner, when the weight or pressure was made three times as large (12 pounds), the friction was found to have been increased in almost the same ratio — to 7.9, according to the actual measurement.

That is, for two given surfaces, the friction depends only upon the pressure between them. If F is the friction, and P is the force pressing the surfaces together, then $\frac{F}{P}$ is constant.

The ratio $\frac{F}{P}$ is called the **coefficient of friction** for the given surfaces. This idea is used so much in practice that the coefficient of friction is usually represented in formulæ by a special symbol, — sometimes the Greek letter ϕ and sometimes μ. For example, according to our measurements the coefficient of friction between the given surfaces of Oregon pine and cedar as used in the experiment would be

$$\phi = \frac{3.8}{5.6} = 0.68 \text{ for the first measurement.}$$

$$\phi = \frac{2.6}{4} = 0.65 \text{ for the second measurement.}$$

$$\phi = \frac{5.25}{8} = 0.66 \text{ for the third measurement.}$$

$$\phi = \frac{7.9}{12} = 0.66 \text{ for the fourth measurement.}$$

$$\text{Average } \phi = \overline{0.66}$$

When we say that the coefficient of friction between the given surfaces of pine and cedar is 0.66, we mean that the total friction between them is always that fraction of the total pressure.

If the pressure acts in a direction not perpendicular to the two surfaces, as by pressing a stick obliquely against the block after the manner shown in Fig. 126, then it becomes necessary to resolve the pressure into two components. Only that part which acts in a direction perpendicular to the surface of contact is taken into account in considering the friction; the component parallel to the surface of contact will tend to push the block along the plane, but it does not thereby change the friction one way or the other.

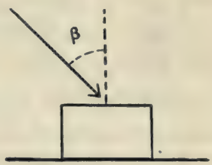


FIG. 126

EXAMPLE

In the following table, containing the values found in the preceding experiments, insert in the column headed "φ" the value of the coefficient of friction for each pair of surfaces named in the same horizontal line.

SURFACES		P	F	φ
I	II			
Brick (On ground face)	Cedar (planed)	5.6	2.5	
Oregon pine (Planed, with weights added)	Cedar (planed)	5.6	3.8	
Brick (On ground edge)	Cedar (planed)	5.6	2.5	
Cast iron (Dressed on shaper)	Cedar (planed)	8.7	4.5	
Cast iron (Dressed on shaper)	Leather (Old belting, flesh side)	8.7	1.8	
Cast iron (Dressed on shaper)	Leather (Same piece, hair side)	8.7	1.5	
Oregon pine	Cedar (planed)	4	2.6	
Oregon pine	Cedar (planed)	8	5.25	
Oregon pine	Cedar (planed)	12	7.9	

In all our illustrations, for the sake of simplicity, we have considered the pressure between the surfaces as being due to the weight of an object. But **the laws of friction apply to pressures of all kinds**, as in a brake operated by levers or by a spring. As a further example, in dealing with the friction in the bearings of a line of shafting we have to consider not only the downward pressure due to the dead weight of the shafting and pulleys, but also the force exerted by the tension of the belt (which may be in any direction); it is the resultant of the two that gives the total pressure of the shaft in the bearings — that pressure from which we must compute the friction.

The use of sliding blocks for our purposes of illustration may also tend to another misunderstanding. We are not dragging the weight of the block when we slide it on a *horizontal* surface — at least not in the sense that we overcome the weight in lifting it. The effort of dragging is not due to the weight of the block, except as the weight causes friction at the surface of contact. It should be remembered that if there were no friction and the plane were *perfectly* horizontal, the slightest force would cause the block to move and to keep on faster and faster; of course, the greater the mass of the block and the smaller the force applied to it, the less rapidly would it gain velocity, but, as already stated on page 83, if there were no friction, any force, howsoever small, would produce motion in any mass, howsoever large, on a horizontal plane.

Static Friction and Kinetic Friction. — In any of the preceding observations, if we had watched the reading of the spring balance after the body had commenced to slip, we would have seen that the pull necessary to keep the body moving is less than that necessary to start it. In other words, the coefficient of friction between the two surfaces when they are in relative motion is less than the friction of rest, or the statical coefficient. However, the friction during motion, or kinetic friction, obeys the same laws that

we have already deduced for statical friction, the friction being simply less by a certain amount for the given surfaces, but still proportional to the pressure. This is true, provided the velocity is not too great; at high speeds the coefficient is less, while at very low speeds the statical coefficient and the kinetic coefficient are nearly equal.

Not only is the statical greater than the kinetic friction, but it is also true that the longer the surfaces have remained in quiet contact, the greater the statical coefficient.

Friction always a Resistance. — In discussing the idea of a force (p. 69), it was stated that friction is not an active agent capable of producing motion; in fact, it is not evident until called into play by our effort to make the two surfaces slide in contact with each other. Furthermore, it has no direction of application such as a force always has; when a brick rests on a horizontal surface, the friction is the same, whatever the direction in which we slide the body.

Determination of the Coefficient of Friction by Means of the Limiting Angle, or Angle of Repose. — This simple method of measuring the coefficient of friction is exemplified in the following experiment:

A block of iron weighing 8.7 pounds was placed on the cedar board and the latter gradually inclined until the block began to slip down the grade. The angle of elevation γ, at which the slipping commenced, was measured by applying an ordinary two-foot square in such manner as to determine the lengths OA and AB in Fig. 127. These lengths OA and AB were found to be 20 inches and $10\frac{9}{16}$ inches respectively. The ratio $\frac{AB}{OA} = \tan\gamma = \frac{10.56}{20} = 0.53$. By compar-

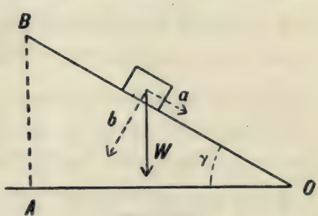


FIG. 127

ing this value 0.53 with the coefficient, 0.52 for the same surfaces (as previously determined by means of the spring balance), it will appear that the coefficient of friction is simply the tangent of the angle at which slipping commenced.

That this is true can be readily shown by simple geometric demonstration. The weight W is a force acting vertically downward, but as the plane of the board is no longer horizontal, only a part of W — the component perpendicular to OB — serves to press the two surfaces together. Accordingly, if W is resolved into two components, — one parallel to OB and the other perpendicular to OB, — the former, component a, tends to drag the weight down the plane, but does not figure in the friction, while the other, b, tends only to cause friction. If ϕ is the coefficient of friction, the total friction is $b\phi$, and if the component a is just sufficient to make the body slip down the plane, then $a = b\phi$. But $a = W \sin \gamma$ and $b = W \cos \gamma$; whence $W \sin \gamma = \phi W \cos \gamma$, or

$$\phi = \frac{W \sin \gamma}{W \cos \gamma} = \tan \gamma, \text{ as was to be shown.}$$

EXAMPLES

1. *A body resting on a surface just begins to slide when the surface is inclined 21° 17′. What is the coefficient of friction?*

2. *A body placed on a 40 per cent grade has just sufficient inclination to cause slipping. What is the coefficient of friction?*

3. *The coefficient between the Oregon pine and the cedar board was 0.66. What is the greatest angle at which the board could be inclined without the block slipping?*

4. *If a body is on an inclined plane, the total friction is less than on a horizontal plane. Why?*

Work Done in Dragging a Body by Sliding. — When a body is dragged with a uniform velocity along a horizontal surface, the only work done is in overcoming the frictional

resistance (see p. 83). If the body is on an inclined plane, the total friction is less than on a horizontal plane; but there is the additional consideration that extra work will be required in elevating the weight, if it is moved up the incline. If the body is being moved down the grade, a part of its weight then aids in overcoming the friction.

EXAMPLES

1. *Resolving the weight into two components a and b, parallel and perpendicular to the plane (as in Fig. 127), and calling the coefficient of friction* ϕ*, what will represent the force necessary to move the body up the plane? What to move the body down the plane? In the latter question what is signified if a is greater than* $b\phi$*? If* $a = b\phi$*, what is the value of* γ*?*

2. *A mass weighing 200 pounds rests on a horizontal surface. If the coefficient of friction is 0.23, what force will be required to drag the body along the surface with a uniform velocity? What horse power is required to drag this body at the rate of 100 yards a minute?*

3. *A body weighing 90 pounds rests on a surface inclined at an angle* $\gamma = 14° \ 3'$*. The coefficient of friction between the two surfaces is 0.375.*

(a) *What force is necessary to drag the body up the plane?*

(b) *What force is necessary to drag it down the plane?*

(c) *What horse power will be required in each case to keep the weight moving at the rate of 40 feet per second?*

The work done in overcoming the sliding friction at the axle of a wagon wheel as the vehicle advances depends on the diameter of the wheel and also upon the diameter of the axle If the friction is 10 pounds and the diameter of the axle is 2 inches, then the work done in a single revolution will be $\frac{3.1416 \times 2}{12} \times 10$, or 5.24 foot-pounds. And if the diameters of the front and hind wheels are 3 feet and 4 feet respectively, then in traveling a given distance the front wheel will

make four thirds as many revolutions as the hind wheel, and hence the energy consumed in a given time will be proportionately greater for the front axle. The same idea applies to the bearings of a railroad car, although in this case the wheel is fixed to the axle, and the journal of the axle turns in the axle box. The relative motion is the same whether it is the wheel or the axle that moves.

Instead of regarding this as a question of energy consumed, it will be readily seen that it could have been treated by the principle of moments.

Likewise, in a line of shafting, the diameter of the shaft must be taken into account in determining the rate at which the friction in the bearings is overcome. The same consideration applies to an end-bearing vertical shaft and to the collars that take the end thrust on a horizontal shaft. The thrust box on a propeller shaft is another good example.

In a watch, friction is reduced by making the ends of the spindles cone-shaped and restricting the bearing part to a very small area near the point.

Rolling Friction. — It has already been stated ("Kinematics," p. 41) that when a wheel rolls along the ground that part of the wheel in contact with the ground is always at rest; as each point comes down and touches the ground that point on the wheel is for the instant at rest relatively to the surface upon which the wheel rolls. There is no sliding of the wheel bodily along the ground, yet there is a resistance, which in time will bring the wheel to rest.

It is this resistance that is called **rolling friction.** As the successive parts of the circumference come in contact with the ground it is not difficult to conceive how the minute projecting particles of the wheel collide with similar elements of roughness on the ground, producing this resistance. The rougher the surface the greater the rolling friction, as we know from the fact that a ball will roll

farther on a smooth floor than on a carpet or on a gravel path. If the surface of the wheel is soft, it will be flattened and its progress retarded. Likewise, if the surface upon which a rolling body moves is such that a depression is formed by the weight of the body, a corresponding ridge will also be formed in the path of the object, so that the energy of the rolling body will be used up in deforming the surface and constantly climbing this little ridge.

If a wheel or ball rolls in contact with a curved surface, the "normal" pressure at any instant would refer to a direction perpendicular to a line or plane tangent in common to the two surfaces at their point of contact for the given instant.

Work Done in Dragging Vehicles. — It has already been stated that when a rolling vehicle is moved, the tractional resistance includes all friction, whether sliding or rolling, both in the bearings and at the perimeters of the wheels. This resistance may be designated as so many pounds to each ton of load, including the weight of the vehicle, or it may be expressed as a percentage of the total load. The latter is called the "coefficient of traction."

EXAMPLES

1. *A train is made up of 12 cars each representing a total load of 23 tons. If the tractional resistance is 10 pounds per ton load, what force will be necessary to keep the train moving with a uniform velocity?*

2. *What horse power will be required to keep this train moving at the rate of 30 miles an hour?*

3. *If this train comes to a 4 per cent grade, what horse power would be required to carry the train up the grade with a velocity of 4 miles an hour?*

Consider that the friction is the same on the slope as on the level.

4. *Why are the shafts of a vehicle placed at an angle slightly above the horizontal?*

Anti-friction Wheels and Ball Bearings. — When a wheel supports a vehicle of any sort in such a manner as to require bearings, the resistance encountered in moving the vehicle includes not only the rolling friction at the circumference, but also some sliding friction in the bearings. To get rid of this sliding friction, or rather to reduce it to a minimum, a number of devices, such as antifriction wheels, roller bearings, and ball bearings, are used under varying conditions.

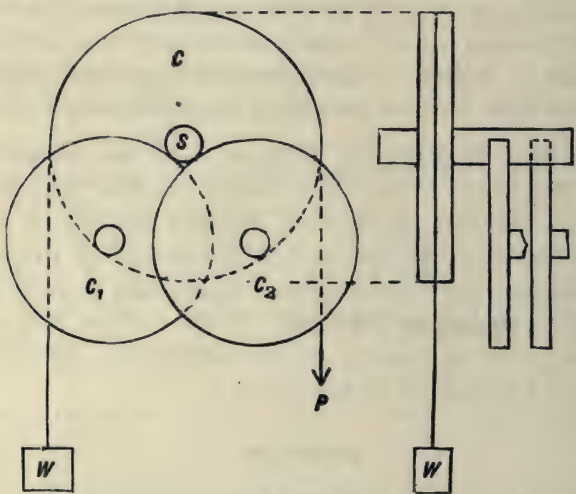


FIG. 128

The manner of using antifriction wheels is shown in Fig. 128. Suppose that a weight W is carried on the circumference of a wheel C. The axle S of this wheel, instead of revolving in fixed bearings, is supported by the rims of two other wheels, C_1 and C_2, the axles of these, however, being in fixed bearings. As S revolves C_1 and C_2 also revolve, but with a much smaller angular velocity. Now the total load and the friction in the bearings of C_1 and C_2 may even be greater than would have existed in the bearings of S if fixed bearings had been used at that point instead of the anti-

friction wheels. But the rate at which the axles of C_1 and C_2 move in their bearings is so slow that very little work is done during each revolution of S, and hence it happens that the energy consumed by sliding friction in the bearings of C_1 and C_2, *plus* the rolling friction of S on the rims of C_1 and C_2, is less than what would have been used up by the sliding friction of S placed directly in fixed bearings.

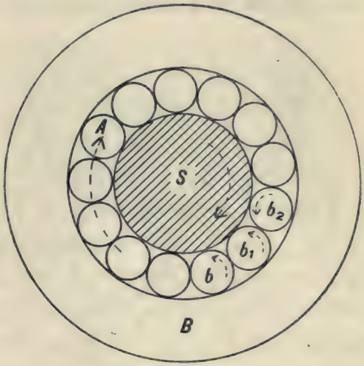


Fig. 129

Ball bearings have become so well known through their extensive application to bicycles that the illustration in Fig. 129 needs no special description. If the axle S revolves in the direction of the hands of a clock, the balls will all revolve in a counter-clockwise direction, and will also progress in a train as suggested by the large arrow A, rolling on the inner surface of the fixed bearing, represented by the portion B. From this it will be seen that where the balls touch S and B, the friction is entirely rolling friction. But if we consider the points of contact between the balls themselves, it will be observed that at all these points there is sliding friction. For example, in Fig. 130, which represents on a larger scale the adjacent balls b and b_1 of Fig. 129, the point p of ball b is shown to be moving upward, and at the same instant the point p_1 of ball b_1 is necessarily moving downward, if the two balls are revolving in the same direction.

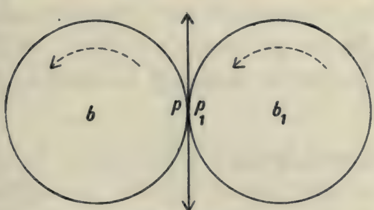


Fig. 130

The occurrence of this sliding at each point of contact between the balls is sometimes urged as an argument against the use of ball bearings, notwithstanding that their great efficiency has been so widely demonstrated in practice. While it is true that the balls must slide in contact with each other, it is also true that there is no great pressure between them, because the weight on the axle S tends to spread the balls outward in the direction of the radial lines shown in Fig. 131, thus tending to separate them from each other in the manner illustrated by the small dotted circles in the figure. Hence, since the pressure at the points of contact is comparatively small, the friction between the balls as they slide past each other cannot be great.

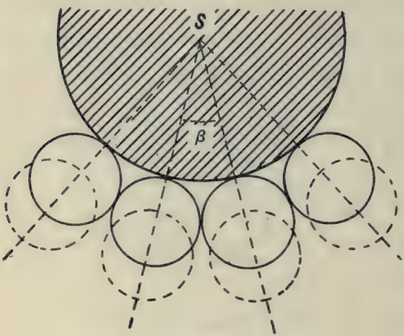


FIG. 131

This sliding friction, though small, exists at every point of contact, so that the greater the number of balls used the greater the total friction. Hence the advantage of using a small number of large balls, rather than a larger number of smaller ones. The larger balls also have the advantage that their tendency to be pressed away from each other is greater because of the greater angle β, indicated in Fig. 131.

Cylindrical and Conical Roller Bearings, while not as common as ball bearings, have come into extensive use — the cylindrical form for journals and the conical form for thrust bearings.

Lubricated Surfaces depart widely from the ordinary laws of sliding friction. — When two surfaces, as in shaft bear-

ings, are separated by a lubricant, such as an oil, a grease, or plumbago, the simple laws of sliding friction are greatly modified, especially as regards kinetic friction, or friction of motion. The friction is no longer between the given solid surfaces entirely, but much depends (1) upon the lubricant itself, — the kind and quantity, and the manner in which it is applied; and (2) upon the relative velocity of the two surfaces, and the pressure between them. (3) A temperature change at the two surfaces is also sufficient to change the entire relations otherwise established.

Notice that these are not simple changes in the values of constant coefficients, but are radical changes of fundamental laws, giving rise to variable coefficients for the same surfaces. At first thought one would say that if the lubricant adheres to each of the two rubbing surfaces, the motion must be a sliding of one part of the lubricant on another part, and hence it would be sufficient to substitute for the coefficient of friction between the given surfaces the coefficient of oil on oil, or plumbago on plumbago, etc. This idea, however, is not upheld by careful observations that have been made.

Scientific investigation has not furnished altogether conclusive results as to the laws that apply to the friction of lubricated surfaces. It is certain, however, that the friction under these circumstances is not a constant ratio of the pressure between the surfaces, as in the absence of lubrication. This *ratio* (the coefficient of friction) for lubricated surfaces decreases as the pressure increases and continues to do so up to a certain limit, beyond which it becomes greater again up to the time "cutting" or abrasion begins.

For unlubricated surfaces the difference between static and kinetic friction (friction to start the motion as contrasted with friction during the motion) is never very great. Between lubricated surfaces the friction at starting is much greater — until the motion of the surfaces has carried the lubricant to the bearings from which it was squeezed out

during rest. This is especially the case in the bearings of heavy machinery, or where the lubricant is unduly limpid.

The controlling conditions — kind of lubricant, its quantity and manner of application, temperature, velocity, and pressure in bearings — are interrelated in such a complex manner that it is practically impossible to express those relations in any but a very general way or with more than approximate correctness.

In general, the lubricant should adhere to the surfaces sufficiently to be constantly dragged in to the rubbing area as fast as needed. It can be forced in by external pressure, under some circumstances.

If the pressure between the bearings is very great, the lubricant may be forced out, especially if it is too thin, and in such cases it is better to use grease or plumbago, or "heavy" oil. In small bearings, under light pressure, a thick, viscous oil would be needlessly cohesive.

Likewise, at varying velocities the amount of lubricant drawn into the rubbing area by the motion of the bearings and the general effect are very variable. At moderate speeds the coefficient of friction decreases as the velocity increases, but for very high speeds the opposite is true. And furthermore, the speed at which this change from a decreasing to an increasing coefficient takes place is different for high and low pressures, and for high and low temperatures, and for different lubricants.

Conditions that are satisfactory at one temperature might be actually reversed at even a slightly different temperature. The properties of oils change greatly under the influence of heat, and no two oils change in the same manner or degree. In general, the lighter oils used for high speeds and low pressures give better service at comparatively high temperatures, while the more viscous oils and greases used for high pressures lose their desired viscosity when heated. If the temperature becomes very high, the organic (animal and

vegetable) oils may be decomposed, liberating injurious acid compounds that attack the material of the bearings. In steam chests and cylinders, oils of only or mainly mineral composition, which are not decomposed by heat, should be used.

On the whole the friction of lubricated surfaces presents many questions for scientific investigation, and it is not safe to draw very broad conclusions from the superficial and limited observations that are likely to be met with in this connection in ordinary practice.

Friction of Ropes, Belts, and Cables. — When a flexible belt or cord is wrapped upon a cylindrical surface, any tension exerted on the belt or cord produces a normal pressure against the cylindrical surface at every point in the arc of contact. This obviously will cause friction, and while this friction will be in accordance with the general laws already deduced, it will involve mathematical considerations of some difficulty. The hitherto simple law of sliding friction ($F = \phi P$), when extended to cover the case of belts and ropes on pulleys and sheaves, becomes so complex that it cannot be expressed without the use of exponential or logarithmic formulæ.

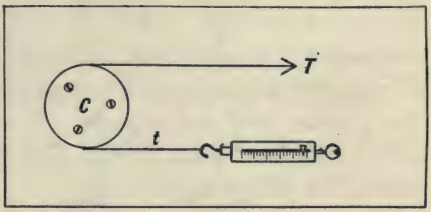


Fig. 132

Suppose that the circle C, in Fig. 132, represents a cylinder screwed firmly to a board so that it *cannot rotate.** A cord

* The transmission of power by means of belts and ropes presupposes that the pulley moves with the belt — a sort of rolling contact. Some fundamental conceptions, however, can be grasped more readily by first regarding the pulley as immovable. In fact, the maximum power transmissible is determined by the conditions at the time of slipping. If the pulley moves with the belt, the coefficient is one of static friction; if the belt is intended to slip on the pulley, as in the friction brake, the coefficient is for kinetic friction. The relations above deduced are the same in both cases, the only difference being in the magnitudes of the constant coefficients for static and kinetic conditions.

is placed around the cylinder in the manner shown, affording a contact for a half circumference, or 180°. One end leads to a spring balance fastened to the board. Now if a tension T be cautiously exerted on the free end of the cord (avoiding sudden jerks), it will be found by reading the spring balance that the tension t in the other part of the rope is much less than T. If the tension T be increased, t will also be found to increase in the same proportion. That is, if the two ends of the cord are kept parallel (so as not to change the length of the arc of contact), the ratio $\frac{t}{T}$ will remain constant; t will always be a definite portion of the applied tension T. Small t is less than T because of the friction of the rope or belt on the cylinder, and this friction is obviously equal to the difference $T - t$. For example, if $T = 5$ pounds, and $t = 1$ pound, the friction of the cord on the cylinder is $F = 4$ pounds. If we increase T to 15 pounds, then t will be 3 pounds and $F = T - t = 12$ pounds. Expressed in different form, if

$$\frac{t}{T} = \frac{1}{5}, \text{ then } \frac{F}{T} = \frac{4}{5}, \text{ or } F = \frac{4}{5} T.$$

Now it is this fact — that the friction is itself a function of the force that overcomes it — which gives rise to the complex mathematical relations that have to be dealt with in studying the friction of belts and ropes. In the simple phenomenon of sliding friction, the force which overcomes the friction between the weight W (Fig. 133) and the surface upon which it rests has no tendency to change the magnitude of this friction one way or the other. In the case of the friction of the rope on the cylinder, on the contrary, the very tension that is intended to make the cord move is what causes the friction on the cylinder, and the greater the tension the greater the friction. Hence, in practice,

FIG. 133

the more power to be transmitted the tighter the belt, and the wider and thicker it must be, to stand the necessary tension.

The Friction of a Belt depends upon the Arc of Contact. — The friction of a belt on a pulley is not only a function of the applied tension, but depends also on the magnitude of the arc of contact. In Fig. 132 the rope or belt was in contact with one half the circumference of the pulley. If now the end to which the tension is applied has a direction such that the rope is in contact with an arc β, less than the semicircumference, then the ratio $\frac{t}{T}$ is no longer $\frac{1}{5}$, as in the last paragraph, nor does the friction $F = \frac{4}{5}T$. The effect of reducing the arc of contact will be to diminish the friction, leaving a greater proportion of the applied tension T to be propagated along the cord or belt to the spring balance, to cause tension t.

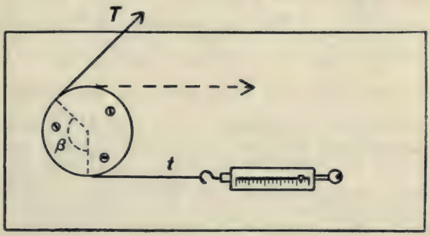


Fig. 134

Hence, the friction depends upon the extent to which the belt envelops the pulley. This is not, as it might appear, in violation of the law asserting that the sliding friction between two objects is independent of the area of contact. The friction between two surfaces is independent of the area of contact, *provided that in changing the area we do not at the same time change the total pressure between the surfaces.* For instance, if a brick is turned from a flat side to an edge, the area of the surface of contact is changed, but the total normal pressure is still the weight of the brick ; the area is less, but the pressure per unit area is greater. But if several bricks are fastened together in a "train," as in

Fig. 135, the pressure per unit area is unchanged, and hence the increased area of contact is accompanied by a proportionate increase of total pressure, thereby multiplying the total friction in the same ratio. The friction of a belt on a pulley is analogous to that of the train of bricks. Every degree added to the arc of contact adds to the total pressure of the belt against the pulley and thus increases the friction.

FIG. 135

Friction of a Flexible Cord completely Enveloping a Cylindrical Surface. — The laws for the friction of belts and ropes apply not only to ordinary transmission of power from pulley to pulley, but also to cases where a rope is coiled several times around a cylinder, as the rope on the drum of a hoisting engine or a ship's hawser around a post. If there is but one turn of the rope around the cylinder, the arc of contact is 360°, for two turns 720°, etc. The tension that must be exerted at the free end to prevent slipping is, as before, the difference between the applied tension T and the friction. With this same applied tension, if we take a second turn of the rope around the cylinder, the value t left after one turn may be regarded as the "applied tension" for the next turn, so that if t_2 is the tension that remains in the free end after two turns, then

$$\frac{t_2}{t}=\frac{t}{T}, \text{ or } \frac{t_2}{T}=\left(\frac{t}{T}\right)^2.$$

If there were three turns of the rope, the tension in the free end t_3 would be found from the relation,

$$\frac{t_3}{t_2}=\frac{t_2}{t}=\frac{t}{T}, \text{ or } \frac{t_3}{T}=\left(\frac{t}{T}\right)^3.$$

For example, suppose that Fig. 136 represents a cylinder of 3-inch or 4-inch diameter fixed so that it cannot turn. The

upper end of a stout cord is hooked onto a sensitive spring balance, and the applied tension is produced by weights. The ratio $\frac{t}{T}$ for one turn is nearly constant for all tensions, but not absolutely so; roughly, suppose that it is equal to $\frac{1}{6}$. Now if we take a second turn around the post, this value, t or $\frac{T}{6}$, must be taken as the applied tension for the second turn, and the tension on the balance will now be (approximately) $\frac{1}{6}t$ or $\frac{1}{36}T$. For three turns the balance will read $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\, T$, or $\frac{1}{216}\, T$.

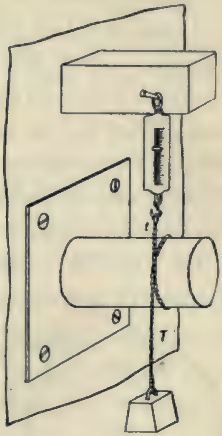


FIG. 136

In general terms, if t_n is the tension left after n turns of the rope, then

$$t_n \propto \frac{T}{K^n},$$

where the exponent n is the number of turns, and K the constant depending on the nature of the surfaces.

This is sufficient to show in a general way that the various factors to be considered in the friction of belts and ropes are related to each other in such a way as to give rise to mathematical expressions of an exponential character. To be made readily usable these exponential expressions must be converted into logarithms. In that way it is possible to express the friction directly in terms of the tension and arc of contact, but this requires the use of the calculus and is too difficult for an elementary text, especially if allowance is made for the centrifugal effect, which at high speeds may materially lessen the friction found by the methods illustrated in Figs. 132, 134, and 136, where the cylinder does not rotate.

Measurement of Power Transmitted by Belts.—In the ordinary continuous transmission of power by means of an endless belt, the angle of contact on each of the two pulleys depends upon their relative diameters and their distance apart. The maximum power that can be transmitted from one pulley to the other through the belt depends on the friction at that pulley where slipping will first occur. If both pulleys are of the same material, this will be at the pulley having the smaller arc of contact.

In Fig. 137, illustrating pulleys of 20-inch and 10-inch diameters respectively, if p_1 is the driving pulley, it will not be possible for the belt to transmit to the second pulley p_2 all the power that the driving pulley would be capable of giving to the belt. The friction of the belt on p_2, in causing the motion of the pulley, is equivalent to a force of the same magnitude acting at any single point on the perimeter of p_2 in a tangential direction. And it should be remembered, again, that this maximum force is not T, the tension in the belt, but only a part of it,—that is, the friction F or $T-t$.

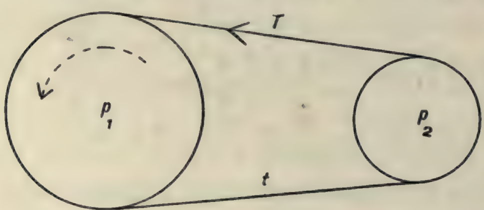


Fig. 137

If the diameter of p_2 is d inches, then for each revolution of p_2 the maximum work that can be done, or the maximum energy that can be taken from the belt by this pulley, is $F \times \frac{\pi d}{12}$ foot-pounds (provided F is in pounds).

If this pulley has n revolutions per minute, the maximum Horse Power that can be transmitted to it from the belt is

$$F \times \frac{\pi d}{12} \times \frac{n}{33000}.$$

EXAMPLES

1. *In the case of the pulleys and endless belt just referred to, if* p_1 *and* p_2 *have diameters of 20 inches and 10 inches respectively, and are 12 feet apart between centers, what is the magnitude of the arc of contact of the belt on each of the pulleys?*

2. *If the shaft of* p_1 *revolves 165 times per minute, what is the maximum power that can be conveyed to the shaft of* p_2, *if the tension in the driving side of the belt is 211 pounds, and the tension on the slack side 40 pounds, when slipping commences? Disregard the centrifugal loss.*

Methods of Increasing the Efficiency of Belts. — Anything that will increase the tension of the belt, or the arc of contact, or the coefficient of friction, will increase the maximum power transmissible by the belt.

We have already seen that the coefficient was increased from 0.4 to 0.45 by covering an iron pulley with leather. The two pulleys should not be near enough together to interfere with the flexible working of the belt. If the pulleys are of unequal size, the arc of contact on the smaller will be greater the farther apart the pulleys. If the slack side of the belt is uppermost, the sag of the belt will increase the arcs of contact.

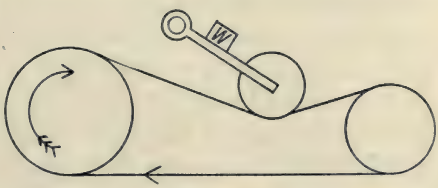


FIG. 138

By the use of a so-called "idle" or tightening pulley, as illustrated in Fig. 138, it is possible to increase both the tension and the arcs of contact at the same time. It is apparent that a device of this sort can be used for vertical as well as for horizontal belts, and that the "idler" can be controlled by a screw or a spring, in place of the weight. If the two pulleys are of unequal size, it is better to have the tightening pulley on the slack side of the belt, near the smaller pulley.

With a little thought it will be seen that the action of any belt necessitates some slipping. One side of the belt being tighter than the other, as each minute length of the belt progresses along the perimeter of the driven pulley to a position of greater tension, it is gradually stretched out, and hence must slip a little. As it again approaches the slack side of the other pulley, the driving pulley, it contracts and slips in the opposite direction. The elasticity of the material, therefore, is a factor to be considered in the easy working of the belt. This constant slipping generates heat, causing a gradual "burning" of the belt.

The friction depends on the tension, the arc of contact, and the coefficient of friction between the two surfaces, but is independent of the width of the belt and the diameter of the pulley. The latter holds true until the lack of perfect flexibility in the belt would prevent it from conforming properly to the surface of too small a pulley.

SECTION III

KINETICS

CHAPTER XI

THE LAWS OF MOTION

Inertia. — Our commonest experience in moving an object is associated with the effort necessary to overcome its weight or to overcome friction due to its weight. Looking into the matter further, however, we find other circumstances that are less obvious, but of which we must now take cognizance. When we speak of the work done in lifting a body, we assume that it is moved with a uniform velocity (see p. 83). Just why we had to make this reservation at that time cannot be told in a few words; once we give the object accelerated, instead of uniform, motion and try to measure experimentally or to calculate mathematically the effort necessary to produce this additional motion, we find ourselves dealing with considerations that engaged the attention of Galileo, Huygens, Newton, Descartes, Leibnitz, D'Alembert, and their contemporaries for several centuries in their endeavor to come to a common understanding of the concepts of Kinetics.

One of these concepts, a first principle of physics, is the inertness of matter — its inability to move itself. On page 80, speaking of mass as distinguished from weight, it was stated that a given mass would not exhibit weight at all if it were not for the presence of a second attracting mass, while, on the contrary, we cannot conceive of it losing its mass.

According to this, would any effort be required in "putting the shot" or in throwing a baseball at a place where objects have no weight; for instance, within a hollow spherical planet, as illustrated in Fig. 45, page 77?

(1) If the shot is hurled horizontally, then to give it a certain velocity the same effort will be required in the one case as in the other. That is, with the same effort the shot will leave the hand with the same velocity (if thrown horizontally) whether hurled from a point on the earth's surface or at a place where mass is conceived to be without weight. In the one case, — in the absence of gravitational attraction, frictional resistance, or other disturbing influences, — the body would keep on moving indefinitely in the direction it had when it left the hand, and with the same velocity. This we infer from the inability of the body to move itself. In the other case, at the earth's surface, the body will not keep on horizontally, but will fall, as we know, in the manner shown in Fig. 37, page 60.

(2) If weight did not exist, the shot could be hurled upward as easily as horizontally, and, whatever the direction, the effort or force exerted would be proportional to the mass and to the velocity which we wish to impart to it in a given time.

These considerations impress upon us the idea that any mass, regardless of its weight but by virtue of this circumstance which we call its inertness, requires effort to give it motion. The greater the mass the greater the effort; then, since the effort is necessary because of the inertness of the mass, it follows that the amount of inertness is determined by the mass, or *vice versa*. Following this train of thought we come to regard "amount of inertness" as a mathematical, measurable entity, and as such it is called **inertia**. Inertia, however, like friction, is not evident until called into play by the very force that it opposes. In the early history of mechanics, and in its subsequent evolution to its present

status, the idea of inertia has had a prominent place; but for us its immediate importance is its usefulness as an aid to the understanding of mass as that attribute of matter through which we interpret force effect in terms of the motion produced by it.

While the idea of inertia enables us to identify mass by means of something quite apart from weight, nevertheless it must be remembered that our commonest perception of mass is through the ever present property of weight, and in our minds this must be constantly contended with as tending to creep into our considerations when it should be kept out. Yet there are times when weight must be duly considered, as when a body falls "by its own weight," obeying these same laws of motion that we are about to develop. We must learn to regard it or disregard it, as circumstances may require.

Furthermore, we must not be misled by the fact that weight and mass are sometimes expressed by means of units of the same name. A 10-pound mass will weigh 10 pounds at a certain designated place on the earth's surface, — at sea level in latitude 45°. It weighs more in polar and less in equatorial regions. If we say that a given body weighs 10 pounds, we mean at the standard place, unless otherwise designated. From that we know that its mass is also 10 pounds. Even when the idea of mass is all that we are concerned with for the time being, it is frequently more convenient to identify the body by merely telling what it weighs.

Momentum or "Mass-Velocity." — By the momentum of a body we mean the numerical product of its mass and velocity — not weight and velocity. It is sometimes spoken of as "quantity of motion." There are no simple names by which one unit of momentum may be distinguished from another; it is customary to state the number of units and leave the magnitude of the unit to be inferred. It may be any unit of mass — pound, ounce, ton, kilogram, gram — combined with any unit of velocity; but an awkward, verbal expression of it is often hard to avoid.

Instead of saying that the momentum is numerically equal to the product of the mass times its velocity, we might have said merely that it is proportional to this product, or $M \propto mv$. Then the unit of momentum could be entirely independent of the units of mass and velocity, and the momentum would be expressed as some multiple or submultiple of mv, as $M = Kmv$.

EXAMPLES

1. *If a body weighing 4 pounds and having a velocity of 5 feet per second is said to have a momentum of 20, what unit of momentum is understood?*

2. *A horse weighs $\frac{1}{2}$ ton and travels at the rate of 10 miles an hour. If we say that his momentum is 5, what unit is implied? Compare his momentum with that of the 4-pound body mentioned in the preceding example.*

3. *Which has the greater momentum, a 10-pound mass moving at the rate of 30 miles per hour or a mass of 5 kilograms having a velocity of 10 meters per second?*

4. *Two bodies have the same momenta. One is a 20-pound mass with a velocity of 4 feet per second. The other is a 2-pound mass; what is its velocity?*

5. *Two bodies have the same momenta. One is a 10-pound mass with a velocity of 50 feet per second. The other has a velocity of 2 miles per hour; what is its mass?*

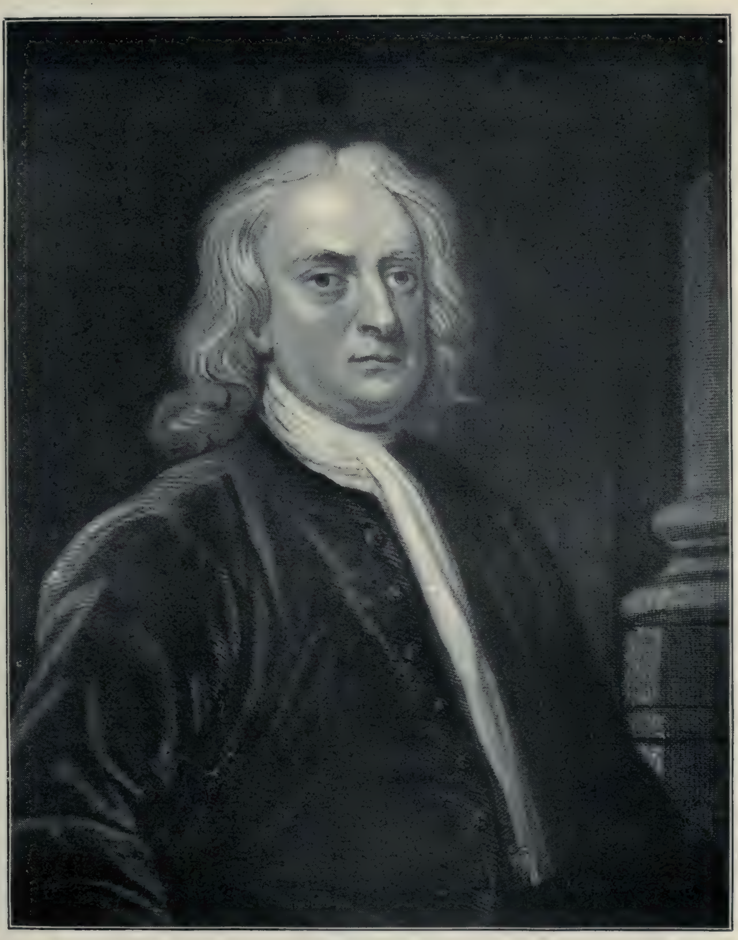
"**Mass-Acceleration.**" — If the speed of a body changes, it gains or loses momentum. This may occur suddenly by collision, or it may take place gradually, either in a spasmodic, irregular manner or as a case of uniformly accelerated motion, illustrated in Fig. 33, page 48.

EXAMPLE

(*a*) *If the body mentioned on page 47 is a 7-pound mass, what momentum did it have at the beginning?*

(*b*) *What was its momentum at the end of 3 minutes?*

(*c*) *At what rate does its momentum change?*



NEWTON

(*d*) *What is its acceleration ?*

(*e*) *Is the product of its mass and acceleration the same as its rate of change of momentum ?*

(*f*) *What will be its momentum at the end of 5 minutes ?*

Algebraically, mv is the momentum of a mass m with velocity v. If its velocity changes from v_0 to v_1, then its change of momentum is $mv_1 - mv_0$, or $m(v_1 - v_0)$. And if this change takes place in t units of time, its average rate of change of momentum is $m\left(\frac{v_1 - v_0}{t}\right)$, and if the change occurred uniformly, this average rate of change of momentum is also its mass acceleration ma.

"Force-Time" and Change of Momentum. — Change of momentum is always the result of some force, and is not only proportional to the magnitude of the force, but also to the length of time it has acted. In general terms $mv \propto Ft$, where v is the velocity acquired by mass m under the influence of a force F acting for t units of time. For practical purposes this relation $mv \propto Ft$ must be put in the form of an equation, which can be done by selecting appropriate units. If we take the ordinary commercial units of the English system, we find that a force of one pound acting for one second on a one-pound mass (that is, a mass that will weigh a pound at the standard place) will give it a velocity of 32.2 feet per second; in two seconds it will give it twice that velocity. Algebraically,

$$v = 32.2\,\frac{Ft}{m}, \text{ or } mv = 32.2\,Ft. \qquad (11)$$

The former of these two equations is interpreted to mean that the velocity generated is directly proportional to the force, directly proportional to the time, and inversely proportional to the mass; the second equation conveys the idea that the change of momentum is a certain multiple of the "force-time." The momentum is the same whether the mass be great or small; but the larger the mass the less the

velocity produced by a given force acting for a stated length of time. With the same effort a person gives a greater velocity to a 9-ounce baseball than to a 16-pound shot.

What we have called the "force-time" is frequently called the "impulse" of the force.

It will be noticed that the constant 32.2 is the same as the numerical value of the acceleration of a falling body, as used heretofore. This is due to the fact that a falling body that weighs one pound has a mass of one pound to be moved and a force of one pound to move it. Substituting these values — one-pound force and one-pound mass — in the equation $mv = 32.2\ Ft$, we get $v = 32.2\ t$, or the acceleration $\frac{v}{t} = 32.2$. If the body weighed ten pounds, we would have $m = 10$ and $F = 10$, or $10\ v = 32.2 \times 10\ t$, or $v = 32.2\ t$, as before. Owing to this fact that the mass and force increase in the same ratio, the acceleration of falling bodies is the same for all masses at a given place. Moved to a place where gravitational attraction is less the mass of a given body remains unchanged, while the moving force F, and hence the acceleration, is less. This constancy of the relation between mass and weight at a given place — the same fact that blinds us to the very idea of mass — diminishes the usefulness of experiments with falling bodies for the purpose of illustrating the laws of motion. It will be better to select our problems under conditions that will permit us to assume forces and masses of any numerical value, without restriction.

If a person "putting" a 16-pound shot exerts a force of 100 pounds in a horizontal direction for two tenths of a second, the shot will leave his hand with the velocity of 40 feet per second. (Verify this result.) As the force then ceases to act, the shot will gain no more horizontal velocity. Its own weight will cause it to fall to the earth, but the time of falling will be the same as if it were merely dropped, and meanwhile it will keep on moving horizontally with the same velocity it had when it left the hand, provided, of course, that air resistance is neglected. Assúming that the vertical

distance of the hand from the ground is 5 feet, then from Formula 6, page 51, it is seen that the shot will reach the ground in $t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 5}{32.2}}$, or about 0.6 second, and meanwhile it will travel horizontally 40×0.6, or 24 feet.

EXAMPLES

1. *(a) A force of 5 pounds acting for 2 seconds will produce what momentum?*

(b) What velocity in a 10-pound mass?

(c) Acting for 7 seconds, what velocity will it give to a 20-pound mass?

2. *A person weighing 150 pounds is mounted on a 30-pound bicycle. A second person gives him a start by pushing with a force of 30 pounds for $2\frac{1}{2}$ seconds. If the bicycle could be moved without friction, what total momentum would the rider and bicycle receive? What velocity?*

Disregard the energy that goes into rotation of the wheels.

3. *A 10-pound mass rests on a smooth horizontal table.*

(a) If it is moved by a horizontal force of 3 pounds, and friction be disregarded, how long before it will have a velocity of 165 feet per second?

(b) How far will it have traveled meanwhile?

(c) How far does it travel during the first three seconds?

4. *A 9-ounce baseball is thrown horizontally with a velocity of 150 feet per second.*

(a) What "force-time" was used in throwing it?

(b) If one tenth of a second was consumed in the motion of throwing, what average force was exerted against the ball, and how far did the hand move?

(c) If it was thrown from a tower 75 feet high, when and where did it strike?

5. *A 15-pound mass rests on a horizontal surface, along which it can be moved without friction. To it a second mass of 5 pounds is attached by means of a cord passing over a*

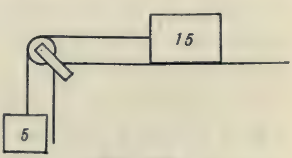


FIG. 139

frictionless pulley, as shown in Fig. 139. By this device a force of 5 pounds is made to move a mass of 20 pounds, provided the mass of the cord and pulley be disregarded,

(*a*) *At what rate will these two masses gain velocity?*

(*b*) *In this arrangement does the 5-pound overcome a weight of 15 pounds? By means of pulleys or other suitable mechanical devices such a thing is possible. How do the two cases differ?*

6. *Two equal masses M and M, each weighing 2 pounds, are arranged as shown in Fig. 140. By means of "friction wheels" (illustrated in Fig. 128, page 176) the friction is reduced to a negligible quantity. Disregard, also, the effort necessary to set up motion in the wheels themselves. If a 4-ounce weight m is placed on one of the equal weights, the entire system will be set in motion.*

(*a*) *What will be the acceleration?*

(*b*) *How far will the overbalanced side fall in 2 seconds?*

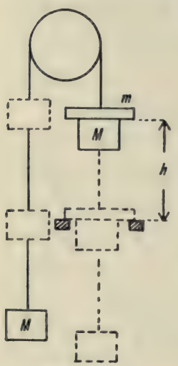


Fig. 140

If the mass m is shaped so as to project beyond M, it is easy to imagine a contrivance by which the extra weight can be left at any desired altitude, while the larger mass continues. The place at which m is to be left behind is distant h feet, let us say, below the starting point.

(*c*) *For the values of M and m given above, what must be the distance h in order that m may be left behind one second after starting?*

(*d*) *If m is left behind 3 seconds after starting, what velocity will the masses M have thereafter? Why will they not gain any more velocity?*

Example 6 is suggested by a classical piece of laboratory apparatus called Atwood's machine, which has long been used to illustrate the laws of falling bodies. Like the assumption of a "frictionless, horizontal plane" in our calculations, this obviates in a practical manner the necessity of overcoming weight, leaving only inertia to absorb the force effect. It also affords the

means of reducing the acceleration from 32.2 feet per second per second to values that are convenient for experimental purposes.

7. *A 7-pound mass is moved along a horizontal plane by a force of 5 pounds against a frictional resistance of 3 pounds. What will be the acceleration?*

HINT. — A force of 3 pounds would be just sufficient to overcome the friction and keep the body moving at a uniform velocity. The force effective in producing motion is what remains.

8. *At what rate will momentum be generated in the system shown in Fig. 141? What will be the velocity at the end of 4 seconds? Numbers in diagram signify pounds.*

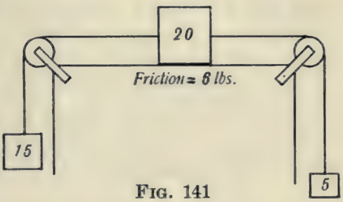


FIG. 141

Force and Kinetic Energy. — When a gun is discharged, the force that impels the bullet is the pressure of gases formed in the explosion. If there were means of measuring this pressure and of determining the time required for the bullet to travel the length of the barrel, from these the momentum acquired by the bullet could be computed, and hence the velocity with which it leaves the gun. With present appliances, however, such measurements would be extremely difficult, if not impossible, within reasonable requirements of accuracy. Instead of calculating the velocity in that way, it is customary to determine it directly by means of a device called the ballistic pendulum — a heavy, suspended block into which the bullet is fired at short range. For larger projectiles the suspended mass is a box containing several tons of sand. Knowing the masses of the pendulum and the projectile, the striking velocity of the latter can be determined by observing the angle through which the pendulum is made to swing by the impact. These computations would be made through the laws of impact (Chap. XIV) and of pendulums (Chap. XIII). However, for our present

purposes, it is desirable to assume certain conditions and values (some of them impossible in practice) that will afford a concrete example with which to illustrate the effect of a force in producing kinetic energy as contrasted with its effect in producing momentum.

The Gun-and-bullet Problem. — A cartridge slug weighing one fourth ounce is fired from a 10-pound rifle. Assume that the pressure due to the expansion of the gases remains constant and equal to 800 pounds per square inch. If the bore of the gun measures 0.075 square inch, the gun and the bullet are pushed in opposite directions by a force of 60 pounds. Now suppose the bullet leaves the muzzle of the gun 0.006 second after firing.

(*a*) What is the velocity of the bullet as it leaves the barrel?

(*b*) What is the velocity of the gun at the same instant?

(*c*) What is the momentum of each?

Verify the following results:

(*d*) Average velocity of bullet while in the barrel = 370.8 feet per sec.

(*e*) Distance traveled by bullet before leaving barrel = 2.225 feet.

(*f*) Average velocity of gun during same time = 0.58 foot per second.

(*g*) Distance traveled by gun during same time = 0.0035 foot.

(*h*) Total length of barrel (less powder space) = 2.2285 feet.

In Fig. 142, d_1 represents the distance, 0.0035 foot, moved by the gun in the preceding problem, and d the distance moved by the bullet, 2.225 feet. The work done by the expanding gases in moving the gun will therefore be 60×0.0035, or 0.21 foot-pound. The work done on the bullet will be 60×2.225, or 133.5 foot-pounds. In other words,

chemical, potential energy of the gunpowder has been transferred to the gun and bullet in the form of mechanical, kinetic energy — 133.5 foot-pounds to the bullet and 0.21 foot-pound to the gun. It is a striking fact that the kinetic

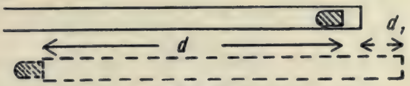


FIG. 142

energy of the bullet is so much greater than that of the gun, while the momentum is the same for both. There was a time when these two notions were regarded as contradictory, nor were they fully reconciled until after a keen controversy in which the greatest mathematicians and physicists of the seventeenth century participated.

The energy imparted to the mass is determined by the "force-distance," $F \times d$; the momentum generated is determined by "force-time," $F \times t$.

The same force acting on two different masses (*e.g.* gun and bullet) for the **same** length of **time** gives the **same momentum** to both.

The same force acting on two different masses for the **same distance** gives them the **same kinetic energy.**

In the "gun-and-bullet" problem it is necessarily the time interval that is the same for both masses. With the same momentum the bullet will have the greater velocity, in inverse proportion to the mass, from the instant it began to move. Hence the force acted on it for a greater distance than on the gun and imparted to it more energy. The given conditions of the problem, however, were force and time, and for that reason we deal first with the question of momentum, using Formula 11, page 193, before we compute the energy.

On the other hand, suppose we had a problem like this: A person gives a bicyclist a start by running behind and pushing uniformly for a distance of 50 feet, thereby giving him a velocity of 20 feet per second. The bicycle and rider

together weigh 200 pounds. With what force was he pushed, disregarding, as before, the energy that goes into the rotating parts of the bicycle?

For this problem we cannot use directly the momentum formula $Ft = \frac{mv}{32.2}$, because we know, not the time interval t during which the force acted, but the distance d. But we can eliminate t in Formula 11 in favor of d. Since the velocity was increased uniformly from 0 to v, the average velocity was $\frac{v}{2}$. The distance $d = \frac{v \times t}{2}$, whence $t = \frac{2d}{v}$. Substituting this value for t in the momentum formula, we have $\frac{F \times 2d}{v} = \frac{mv}{32.2}$, or

$$Fd = \frac{mv^2}{2 \times 32.2}. \tag{12}$$

Now the bicycle problem can be solved by substituting in Formula 12 the values $d = 50$, $m = 200$, and $v = 20$.

Since Fd is the work done by the force that produced the velocity v in mass m, it follows that the second member of Equation 12 represents the kinetic energy of the body. (Remember that the unit of energy in this case is a foot-pound.) In this way we have proved again what was shown on page 91 — that the kinetic energy of a body is proportional to the square of its velocity. Now we can see, perhaps more clearly than before, why the bullet has more energy than the gun, with the same momentum. Its velocity is 640 times as great, and if its mass were the same, it would have more energy according as the square of 640.

For further illustration, if two masses m and M, Fig. 143, approach each other because of a mutual attraction A, they

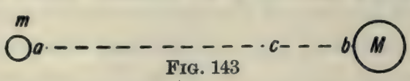


Fig. 143

will gain momentum at the same rate. If $M = 3m$, the velocity of m at any instant is three times that of M, provided.

both started from rest. Having three times the velocity and one third the mass, its kinetic energy will be $3^2 \times \frac{1}{3}$, or three times, that of M. If c is the meeting point, $ac = 3\ bc$, and the work done upon m by the force A is three times that done upon M.

EXAMPLES

1. *A body whose weight is 15 pounds has a velocity of 96.6 feet per second. What is its kinetic energy?*

2. *A body has a velocity of 20 feet per second and weighs 100 pounds. What kinetic energy has it?*

3. *A body weighing 50 pounds and having a velocity of 40 feet per second moves along a horizontal plane against a frictional resistance of 2 pounds.*

   (*a*) *How far will it have traveled before coming to rest?*

   (*b*) *How long will it have traveled before coming to rest?*

4. *A freight car weighing 30 tons is switched onto a horizontal track at the rate of 20 miles per hour.*

   (*a*) *If the total resistance occasioned by the brakes, friction on the track, and air resistance is 2 tons, how far will the car move before it is brought to rest?*

   (*b*) *What momentum had the car at first?*

   (*c*) *At what rate does it lose momentum?*

   (*d*) *How long before it comes to rest?*

It is suggested that most of the examples on pages 94–6 be reviewed at this time.

Absolute Units of Force. — In the English system of measurements the only force with which we have dealt thus far is one pound, with its standard fractions and multiples. A force of one pound acting for one second on a pound mass will give it, not a velocity of one foot per second, but a velocity of 32.2 feet per second. Therefore, by logical analysis, to give it a velocity of *one* foot per second requires a force of $\frac{1}{32.2}$ pound. A force of this magnitude is called a

poundal. The application of this unit, instead of one of the exact fractional parts of a pound (*e.g.* a half ounce, which it closely approximates), is justified by two very important advantages :

First, it greatly simplifies the mathematical calculations necessary in applying the laws of motion. In Formula 11, page 193, it eliminates the constant 32.2. The momentum generated is still proportional to the force-time, or $Ft \propto mv$, but by using the poundal in place of a pound, the equation becomes $Ft = mv$. Dividing both members by t we get

$$F = \frac{mv}{t}, \text{ or } F = m\alpha \qquad (13)$$

where α is the acceleration. In other words, the number of poundals is numerically equal to the total mass-acceleration, or one poundal is numerically equal to a unit of mass acceleration.

Second, it obviates any uncertainty or confusion that may come from having to designate a particular place on the earth's surface at which to measure the acceleration of gravity. The force necessary to generate momentum at unit rate is the same the world over, and for that reason the poundal is called an **absolute unit.** Ordinarily we do our thinking in terms of the so-called **gravitational**, or more properly **weight**, unit — a pound. For research and other advanced work in science the absolute unit is more serviceable. In engineering practice both are used.

In the metric system the gravitational or weight units of force are the ordinary metric weights. A force of one gram acting for one second on a mass of one gram will give it a velocity of about 981 centimeters per second. In other words, the acceleration of falling bodies at Paris is about 981 centimeters per second per second. (Compare this value with the answer of Example 10, page 55.) Algebraically, in weight units $981\ Ft = mv$.

The absolute unit of force in the metric system is the force necessary to generate momentum at unit rate, and hence is $\frac{1}{981}$ of a gram weight, measured under standard conditions. This unit is called a **dyne**, and applies to Equation 13, page 202, like the poundal in the English system. That is, a dyne is numerically equal to a metric unit of mass acceleration.

Instead of reducing the magnitude of the force from a pound to a poundal in order to avoid the presence of a numerical coefficient in the equation $F = ma$, the same result could have been had by retaining the pound as the unit of force and using a larger unit of mass, equal to 32.2 pounds. Logically, there appears to be no preference as to which unit we modify, but practically there are circumstances under which it may be better to forego the advantages of the absolute system mentioned above, and to modify the mass unit rather than the force unit.

If we plan to have the force unit such that, acting for one second on a **mass of one pound**, it will give it a velocity of one foot per second, then the magnitude of this force unit must be $\frac{1}{32.2}$ pound, or one poundal, as already explained. But if we plan to have the mass unit such that a **force of one pound** acting for one second on this mass unit will give it a velocity of one foot per second, then the magnitude of this mass unit must be 32.2 pounds. For this unit of mass there is no name of wide acceptance, although the words "matt" and "ert" have been suggested for the English and metric systems, respectively, and more recently "gee pound" and "gee kilogram."

In the case of a body falling freely, the moving force is the weight of the body W, which varies from place to place. In the equation $32.2\ Ft = mv$, the constant 32.2 was taken in round numbers as the acceleration of falling bodies at the standard place where a pound mass weighs just one pound.

The letter g is used to represent "acceleration of gravity" in general terms, and g_0 represents its particular value at the standard place. Substituting g_0 for 32.2 and W for F, the momentum equation becomes $g_0 Wt = mv$, or $g_0 W = \frac{mv}{t}$. The quotient $\frac{v}{t}$ in this case will be the acceleration of gravity or g at the place where the body happens to be located, whence $g_0 W = mg$, or $W = \frac{mg}{g_0}$. Although this relation was derived from an equation expressed in units of the English **weight** system, the same formula would serve for the metric weight system, the ratio $\frac{g}{g_0}$ being the same whether expressed in English or metric units.

In **absolute** units (pound mass and poundal force, or gram mass and dyne force) the factor g_0 disappears, as in Equation 13, page 202. In this equation the moving force for a falling body is the weight of the body W (in absolute units), and the acceleration is the local g, whence $W = mg$.

Let us interpret again these equations:

$$W = \frac{mg}{g_0}, \text{ in weight units,}$$

and

$$W = mg, \text{ in absolute units.}$$

The former signifies that the weight of a body (in pounds or grams) at any place is equal to its mass (in pounds or grams, as the case may be) multiplied by the ratio of the local acceleration of gravity to the acceleration at sea level, latitude 45°, as a standard. The value of g at sea level varies from 32.091 at the equator to 32.255 at the pole. For latitude 45° we have assumed 32.2, but it is more nearly 32.17. These values represent the acceleration of gravity, corrected for the effect due to the earth's rotation. The symbol g, therefore, is really a local weight

constant, and not a value of the acceleration of gravity. As already explained, however, the expression "acceleration of gravity" is commonly used with this correction implied.

The second equation $W = mg$ signifies that the weight of a body (in poundals or dynes) is equal to its mass (in pounds or grams) multiplied by the local weight constant. Instead of poundals weight and pounds mass, the weight could have been expressed in pounds, provided the mass were in "matts." In fact, the latter combination of units is preferred by many engineers, because it obviates the necessity of introducing into practical work units that may lead to confusion, the poundal being nearly but not exactly equal to a half-ounce weight, and the dyne being nearly, but not exactly, equal to the milligram weight.

Transforming the second equation into $\frac{W}{g} = m$, we can use it to find the mass of a body by dividing its observed weight at any place by the local acceleration of gravity. For this purpose the weight can be observed on a spring balance calibrated at sea level, latitude 45°, but not on a beam balance. (Why?) In the form $\frac{W}{m} = g$ this formula enables us to find the value of g at any place by observing the local weight of a known mass. As the observed weight will be a statical determination, it follows that the equation in this form is not used as a momentum formula.

Absolute Units of Energy. — The foot-poundal is the absolute unit of energy and work in the English system; its relation to the foot-pound is too obvious to need explanation.

The absolute unit of energy and work in the metric system is the **erg**, which is the work done by a force of one dyne when it causes a displacement of one centimeter in its own line.

EXAMPLES

1. *Which is the greater, a dyne or a milligram weight?*

2. *An acceleration of 981 centimeters per second per second is equivalent to how many feet per second per second?*

3. *What name is given to the metric gravitational unit of energy and work? What is its magnitude as compared with an erg?*

4. *A force of 50 poundals acts on a mass of 10 pounds. At the end of 5 seconds what is its velocity? What is its velocity at the end of each of the first 5 seconds? What is its acceleration?*

5. *A force of 50 dynes acts on a mass of 100 grams. What is the velocity of the body at the end of 10 seconds? What acceleration? If the mass had been one kilogram, what would have been the velocity in feet per second at the end of* $3\frac{1}{2}$ *seconds?*

6. *A given body weighs one pound on the earth's surface, radius of the earth 4000 miles. What would be the diameter of a planet (its density the same as the earth's and both assumed to be of uniform density throughout) on which the body would weigh one poundal? What would be the acceleration of gravity on such planet?*

7. *Review Examples 7 and 8, pages 76 and 77. What would be the acceleration of gravity on each of the three planets therein referred to?*

8. *What force, acting for 5 minutes, would give a kilogram mass a velocity of one kilometer per minute?*

9. *How long must a force of 3 poundals act on a 5-pound mass to change its velocity from 10 feet per second to 10 miles per hour?*

10. (*a*) *In Example 1, page 175, what additional force is necessary if the train, having been started from rest, is uniformly accelerated to a velocity of 30 miles an hour in the first mile?*

(*b*) *What additional power is necessary?*

11. *An elevator weighing* $\frac{1}{2}$ *ton starts upward with 10 persons, averaging 150 pounds each.*

(*a*) *If it acquires a velocity of 5 feet per second in the first 6 feet, what force is necessary?*

(*b*) *What power is used?*

(*c*) *After acquiring this velocity, what power is necessary to keep it moving upward with the same velocity?*

Newton's Laws of Motion. — The principles of kinetics were stated by Newton in the form of three laws, with a number of corollaries. They were written in Latin. Even in a liberal translation the language seems quaint and is not altogether in conformity with the modern terminology of mechanics.

His first law covers the idea of inertia.

His second law expresses the relation $Ft \propto mv$.

His third law deals with action and reaction.

Tension in Ropes attached to Accelerated Masses. — A 20-pound mass (Fig. 144) rests on a horizontal plane; coefficient of friction, 0.4. A force of 8 pounds would be just enough to keep the body moving with a uniform velocity. Any excess will produce acceleration.

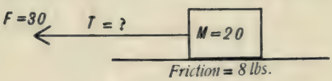


FIG. 144

It is supposed that some way is provided by which the force is maintained at 30 pounds, and that it has to produce acceleration in only the one mass. In that event, the full force is propagated through the rope to the mass, and the tension is 30 pounds.

(Assuming that the friction remains constant as the velocity increases, what will be the acceleration in Fig. 144 ?)

If the force of 30 pounds is supplied by a weight attached to a cord that passes over a frictionless pulley, as shown in Fig. 145, the acceleration will be 14.16 feet per second per second, disregarding always the masses of the cord and pulley. (Verify this.) But consider the tension in the rope — will it be the same as in Fig. 144 preceding? In the preceding case the force had to move only the 20-pound mass; in the present case (Fig. 145) the 30-pound weight has to move its own mass besides. The tension (call it T) is equal in all parts of the rope, and hence pulls back on the vertical mass as much as it pulls forward on the horizontal mass. The accelerating force acting on the vertical mass (considered apart from the other

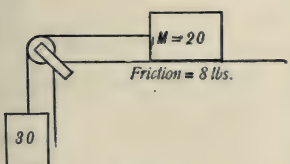


FIG. 145

mass) is, therefore, $30 - T$ pounds, and since the acceleration, as already stated, is 14.16, we get by substituting in Formula 11

$$32.2(30 - T) = 30 \times 14.16. \quad \text{Solving, } T = 16.8 \text{ pounds.}$$

To verify this result, apply the same test to the force acting on the horizontal mass — the rope tension less friction. Its acceleration, of course, is the same as that of the vertical mass. Whence, $32.2(T - 8) = 20 \times 14.16$, or $T = 16.8$ pounds, as before.

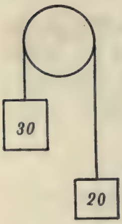


FIG. 146

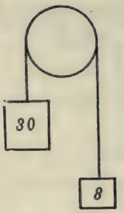


FIG. 147

In Fig. 146 the effective force is 10 pounds for the combined masses of 50 pounds. Assuming a frictionless and weightless pulley, find the acceleration and the rope tension. Contrast the conditions of Fig. 146 with those of Fig. 145.

In like manner, find the acceleration and rope tension for Fig. 147, and contrast with conditions of Fig. 145.

As a general proposition, the rope tension can be found without first getting the acceleration. Let M be the larger mass in Fig. 146, m the smaller mass, and T the tension, all in pounds. The effective force on M alone is a **weight** of M pounds less T, whence $32.2(M - T) = M\alpha$, where α is the acceleration in feet per second per second. The effective force on m alone is $T - m$, whence $32.2(T - m) = m\alpha$. From these two equations eliminate α; thus, in the first $\alpha = \frac{32.2(M - T)}{M}$, and in the second $\alpha = \frac{32.2(T - m)}{m}$. Equating these values of α, $\frac{32.2(M - T)}{M} = \frac{32.2(T - m)}{m}$. Clearing of fractions, dropping parentheses, transposing, and dividing, $T = \frac{2mM}{M + m}$. (Verify this.)

CHAPTER XII

CENTRIFUGAL RESISTANCE

Central Force; Radial Acceleration.—Circular motion being quite as familiar as rectilinear motion, we do not ordinarily think of one as being derived from the other. Yet, if we accept the so-called principle of inertia as a universal truth, it follows that a body moves in a circle only because it has been forced out of a straight line path. The force by which this is accomplished has certain peculiarities which it is the purpose of this chapter to consider. The investigation here, however, will be limited to the case of uniform motion in the circle.

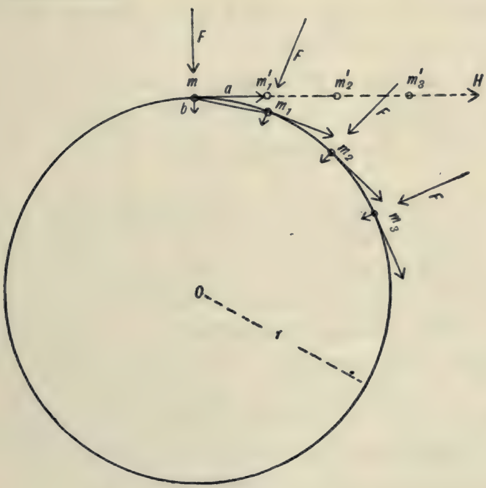


Fig. 148

In Fig. 148 a given mass at m has a uniform velocity in the direction mH. In equal intervals of time it would cover equal distances mm'_1, $m_1'm_2'$, etc. Meanwhile, im-

agine a constant force F acting in such a way as to continuously deflect it into the circumference of a circle described about O as a center, so that it occupies at the end of these successive time intervals the positions m_1, m_2, m_3, etc., its lineal velocity remaining the same that it would have been had it been allowed to continue in the direction mH. A sense of the effort necessary to change the direction of a moving mass in this manner can be had by observing a line of football players swerving under the pressure of an opposing team.

Given this mass of m pounds moving from an initial position m with a uniform velocity of v feet per second in a horizontal direction, what must be the magnitude of the force F which, always acting toward the center, will be just sufficient to deflect the body into this circular path without changing its lineal velocity?

In t seconds the mass would have moved horizontally a distance a, equal to vt feet.

Meanwhile the force F, urging it toward the center, gives it a second component — a uniformly accelerated motion — at right angles to the horizontal component.

In the initial position of m the conditions appear to be those of a projectile falling under the influence of gravity, but it will be found that the two cases have a very important point of difference. In Fig. 148 the two components, while continuing at right angles to each other, do not remain horizontal and vertical, as they do in the case of a projectile. In circular motion the uniform component is always tangent to the curve, while the uniformly accelerated component is always directed toward the center of the circle. Acting together, they give just the right conditions to maintain uniform motion in the circle.

Owing to the ever changing direction of F and v, it is impossible to construct a diagram that will be more than approximately correct. The method of demonstration, also, must take the form of the theory of limits in geometry.

Let v_c be the velocity in feet per second which a force of F poundals would give to the mass of m pounds in direction MK, Fig. 149, if it continued to act in that direction for a brief interval of time, t seconds. Then $Ft = mv_c$, or $v_c = \frac{Ft}{m}\cdot$ Meanwhile, the distance traversed toward the center will be, in feet, $b = \frac{v_c}{2} \times t.$

Notice in this demonstration that a and b are distances, and not velocities. Moving through the distances a and b concurrently, the resultant path would be the chord ML, Fig. 149. If the two components remained horizontal and vertical for any appreciable length of time, the point L in Fig. 149 would not correspond with point m_1 in Fig. 148; but our demonstration will show that this difference will vanish as the time interval approaches zero. In other words, from an inscribed polygon of which one side is ML, of appreciable magnitude, we will reason to a circle as the limit of the polygon when the number of sides is increased indefinitely.

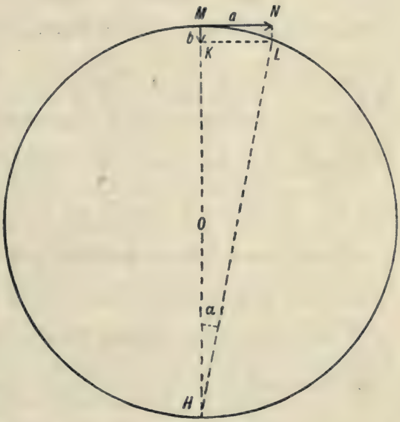


FIG. 149

In the similar triangles MLK and LKH, $MK : LK :: LK : KH$, or $b : a :: a : 2r - b$; whence $b = \frac{a^2}{2r - b}\cdot$ For purposes of construction, we assumed an appreciable length of time in order to get visible magnitudes for a and b, but suppose now that this becomes smaller and smaller. The angle α will then become less and less; the chord ML will tend to coincide with the arc ML; and the value of b will become inappreciable as compared with the finite quantity $2r$. At the limit, therefore, we have $b = \frac{a^2}{2r}\cdot$

But, in terms of the velocities, $b = \frac{v_c t}{2}$ and $a = vt$ as already shown; whence $\frac{v_c t}{2} = \frac{v^2 t^2}{2r}$, or $v_c = \frac{v^2 t}{r}$. Substituting this value of v_c in the equation $Ft = mv_c$, we get $Ft = \frac{mvt^2}{r}$, or $F = \frac{mv^2}{r}$. In other words, the radial force necessary to continuously deflect a body from a rectilinear into a circular path, the speed remaining constant, is directly proportional to the mass of the body, directly proportional to the square of its velocity, and inversely proportional to the radius of the circle.

Corollaries. — I. In the equation $v_c = \frac{v^2 t}{r}$, v_c is the velocity given to the body by the force F in t seconds, whence $\frac{v_c}{t}$, or $\frac{v^2}{r}$ is the acceleration of the body toward the center (which, of course, is also evident from the equation $F = \frac{mv^2}{r}$, because a force is always equal to the mass times the acceleration produced, and hence in this case $\frac{v^2}{r}$ must be the acceleration). *Show that this radial acceleration is constant.*

While there is thus a radial or normal acceleration, there is really no normal component velocity, for the deflecting force constantly changes direction and does not act for a finite length of time in the direction of any one radius. In our demonstration v_c ceases to have a finite value at the limit. It may seem a paradox to say that a body is uniformly accelerated toward the center but never gets any nearer to it. It must be remembered, however, that velocity has both magnitude and direction, and a change of either means acceleration.

II. The force F is equal to $mr\omega^2$, where ω is the angular velocity in radians per second. *Prove this.*

III. If m is expressed in grams, v in centimeters per second, and r in centimeters, then F in the equation $F = \frac{mv^2}{r}$ is in dynes.

IV. In gravitational units $F=\frac{mv^2}{32.2\,r}$ for the English system, and $F=\frac{mv^2}{981\,r}$ for the metric system.

V. The central force F causes no change in the lineal velocity of the body, because, being always perpendicular to the circumference, it has no component in the tangential direction.

EXAMPLES

1. *State the magnitude of each unit in the formula* $F=\frac{mv^2}{r}$ *for the English and metric systems respectively.*

2. (*a*) *What force applied as in Fig. 148 is necessary to maintain in a circular path of 6-foot radius a 10-pound mass having a lineal velocity of 50 feet per second?*

(*b*) *What is the angular velocity of this body in revolutions per minute, and in radians per second?*

(*c*) *Using the formula* $F=mr\omega^2$, *find the value of* F *by substituting for* ω *the angular velocity in radians per second.*

3. *A mass of 500 grams has a velocity of 100 centimeters per second. What normal force will be necessary to deflect it into the circumference of a circle of 20 centimeters radius? Express answer in both dynes and kilograms.*

4. *A body moving in a 2-foot circle under the influence of a central force makes 300 revolutions per minute. If it is a 6-pound mass, find the value of* F.

5. *A 12-ounce block of wood is glued to the rim of a 20-inch pulley, making 150 revolutions per minute. If the face of the wood in contact with the pulley is 3 inches by 5 inches, and the thickness of the block is 2 inches, what must be the strength of the glue in pounds per square inch, to hold?*

6. *A mass weighing one pound is fastened to a string and revolved in a vertical circle of 3-foot radius, making 120 revolutions per minute. What will be the tension in the string when the object is at the bottom of the circle? At the top of the circle? At any position between?*

7. *A boy rotates in a vertical circle a quantity of liquid in an open vessel. If his arm's length is 20 inches, how fast must he rotate the vessel in order that the liquid may not fall out?*

Centrifugal Resistance. — Opposed to the central force F there is always a resistance of the kind that any mass offers to an accelerating force — what was called inertia in the preceding chapter. As fast as the mass changes direction in the circle its inertia is ever ready to serve as a resistance in the new direction. This resistance is commonly called centrifugal "force," and is referred to as the "reaction" of the central force. Both of these expressions are objectionable, if not incorrect. Centrifugal "force" would imply some active agency capable of making the body move outward in a radial direction, whereas we know that it will move in a tangential, and not in a radial, direction whenever it is freed from the influence of the central force. When a stone fastened to a string is whirled around there is tension in the string, but when the string breaks the mass flies tangentially, like the water from a grindstone. We are deceived into thinking that the velocity with which it flies away is due to the outward pull on the string, but the illusion disappears when we reflect that a body moves in a circular path only when forced to do so, and will keep on in a rectilinear path whenever it is let alone.

The way in which these forces and resistances are involved is less likely to be misunderstood when considered in connection with planetary motion. In Fig. 150 let E and M represent the earth and moon respectively. The moon is held in its orbit by the earth's attraction, and would move in a tangential direction MH if this attraction did not exist. If the action of gravity could be instantly suspended, it is clear that there would be no force tending to urge the moon outward from the center — no centrifugal "force." So with the tension of the string tied to the whirling stone: the string itself is a tangible

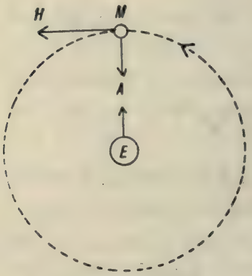


Fig. 150

object; but if we imagine any cross section of it, the force of attraction between adjacent molecules will be quite as invisible as is the gravitational attraction between the earth and the moon.

The arrow drawn from M inward represents the pull of the earth on the moon; the arrow from E outward represents the pull of the moon on the earth. Together they constitute the mutual action, or the two aspects of the attraction, that exists between the two bodies. If we call either one the action, then the other is its reaction. The pull of the earth on the moon is a **centripetal force** (centripetal meaning, literally, "seeking the center"), and its action calls into play a **centrifugal resistance.*** While the centrifugal resistance is equal and opposite to the centripetal force, nevertheless it could not properly be called its reaction — not, at least, according to our definition of "action and reaction."

Just as the moon revolves about the earth, so the earth revolves about the moon; both revolve about their common center of gravity, which is a point in the line of their centers nearer the earth's center in the inverse ratio of the masses of M and E. The centripetal pull of the moon on the earth is in equilibrium with the centrifugal resistance arising from the motion of the earth in this small circle that it describes around the common center of gravity.

Imagine a block of wood resting quietly on the rim of a pulley. If the pulley revolves, the block will be thrown off tangentially. If it is glued to the pulley, the stress in the glue will pull the block inward and will pull outward on the rim. The inward pull will overcome the inertia of the block, while the outward pull will give a thrust of the shaft

* Centrifugal means, literally, "flying from the center," and in that sense is not strictly applicable, the resistance in this instance being purely passive and not capable of making the body fly outward. "Centrifugal resistance," however, is less objectionable than "centrifugal force," and the former we will continue to use in the absence of a term that could be safely substituted for the word "centrifugal."

in its bearing, unless there is a counterbalance on the other side of the pulley, after the manner of the driving wheels of a locomotive.

EXAMPLES

1. *If the mass of the earth is 81 times that of the moon, and their distance apart from center to center is 250,000 miles, locate their common center of gravity.* (See Fig. 70, p. 128.)

2. (*a*) *Having found the distance from the center of the moon to this common center of gravity, find the magnitude of the central force that holds it in its orbit, assuming that it makes one revolution in 28 days.* (If we do not know the mass of the moon, call it M pounds and find the value of the central force F in terms of M.)

(*b*) *In the same manner call the mass of the earth 81 M, and find the central force that holds the earth in its orbit around the common center of gravity (around which, of course, its period of revolution is the same as that of the moon).* (The central force is the same as if the entire mass of the rotating body were at its center of mass.)

Note. — Since the gravitational attraction of the moon on the earth is equal and opposite to that of the earth on the moon, and since these are respectively the forces * that hold the earth and the moon in their orbits, Examples 2 (*a*) and 2 (*b*) should have the same answer.

3. *If the earth were rotating 17 times as fast as it is now rotating, loose objects on the equator would begin to move off into space, because the central force necessary to keep any body in the path of the equator would be greater than that provided by the attraction of gravity. Prove this, assuming a mass of one pound as the object under consideration, and 4000 miles as the earth's radius.* (This refers to the diurnal rotation of the earth, and not to its revolution around the sun, or around the moon.)

* The plural is used, although the two forces referred to are the two aspects of one and the same force, — action and reaction.

4. *Prove that the present rate of the earth's rotation makes the weight of objects at the equator $\frac{1}{289}$, or 0.0035 part less than the gravitational attraction.*

Rotation of the Earth. — When an elevator starts downward, a person standing in it does not exert on the floor of the cage a pressure equal to his full weight — that is, not as long as the velocity of the cage is being accelerated. When the downward motion has become uniform, his full weight is sustained. When an elevator starts upward, the floor has to sustain a load equal to the full weight of the person plus the force necessary to produce his upward acceleration, but is relieved of the extra load as soon as it attains a uniform velocity.

A body resting freely on the earth's surface at the equator loses weight like the person in the elevator during downward acceleration. If we conceive the object to be constantly tending to move tangentially, and its resting place to be a rigid part of the spherical earth, the point of support is constantly accelerated toward the center of the earth and, like the elevator cage, falls away from the body resting on it. The body follows under the influence of gravity, but a part of the gravitational attraction is used up in overcoming the centrifugal resistance of the body, and so the apparent weight of the latter is less than it would have been in the absence of terrestrial rotation.

This apparent loss of weight for objects at the equator, as found in Example 4 above, is about $\frac{1}{289}$ or $\frac{35}{10000}$ of their true weight. At other places on the earth's surface the central acceleration is less in magnitude, and is not directed toward the center of the earth but is always normal to the earth's axis. In Fig. 151 let *NS* represent the earth's axis and *EQ* a radius in the plane of the equator. The angle λ is the latitude of a point *P* on the earth's surface where a given object is located. As

the earth rotates P will describe a circle of radius LP, or a parallel of latitude as it is called in geography.

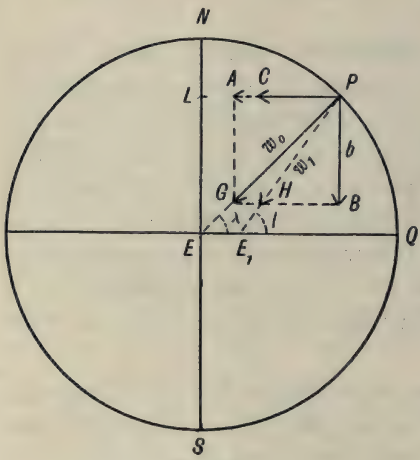


FIG. 151

Let r = the equatorial radius EQ in feet.

Let r_1 = the radius LP, also in feet.

Let w_0 = the gravitational attraction in pounds between the earth and the given body, represented by PG. (This would be the observed weight of the body if the earth were not rotating.)

Resolve PG or w_0 into two components parallel and perpendicular to the equator and

let $a =$ the component PA,

and $b =$ the component PB.

Let $c =$ the central force (represented by CA in the diagram) necessary to maintain the object in the circle of radius LP or r_1.

Then, $a = w_0 \cos \lambda$ and $b = w_0 \sin \lambda$. At the equator the centripetal force $mr\omega^2$ in pounds is $0.0035\,m$, as already shown. At P the angular velocity ω is the same as at the equator, but $r_1 = r \cos \lambda$, whence the centripetal force at P is $c = 0.0035\,m \cos \lambda$. This centripetal force is contributed by the gravitational component a, leaving only $a - c$, represented by PC in the diagram, as the effective component parallel to the equator.

Diminishing the component a in this manner and leaving component b unchanged shows why the earth has been flattened by its own rotation, giving it the form of an oblate spheroid, as it is called, and making the polar radius about 13.5 miles shorter than the equatorial radius.

If we recombine the forces PC and PB, their resultant will give the magnitude of the force which is taken as the observed weight of the body at P, and the direction of the resultant will be that of a plumb line at the same place. If w_1 is the observed weight of the body in pounds, then

$$w_1{}^2 = (a - c)^2 + b^2,$$

or

$$w_1{}^2 = a^2 - 2\,ac + c^2 + b^2.$$

Substituting for a, b, and c the values previously deduced,

$$w_1{}^2 = w_0{}^2 \cos^2\lambda - 0.007\, w_0 m \cos^2\lambda + 0.0035^2 m^2 \cos^2\lambda + w_0{}^2 \sin^2\lambda$$

$$= w_0{}^2 (\cos^2\lambda + \sin^2\lambda) - 0.007\, w_0 m \cos^2\lambda + 0.0035\, m^2 \cos{}^2\lambda.$$

Whence

$$w_1 = \sqrt{w_0{}^2 - 0.007\, w_0 m \cos^2\lambda + 0.0035^2 m^2 \cos^2\lambda.}$$

The third term under the radical sign, since it contains the squares and product of small decimals, is too minute to be of any value in practical measurements and hence may be disregarded. In the second term under the radical sign w_0 is numerically equal to m, according to our notation; both being expressed in pounds, the weight is equal to the mass, because we have assumed that w_0 is the gravitational attraction of m, or its weight, if the earth were not rotating, and because in our diagram we have represented the earth as a perfect sphere. Therefore, omitting the third term and putting w_0 for m in the second term, the equation becomes

$$w_1 = \sqrt{w_0{}^2 - 0.007\, w_0{}^2 \cos^2\lambda,}$$

or

$$w_1 = w_0 \sqrt{1 - 0.007 \cos^2\lambda}. \qquad \text{(a)}$$

In this equation, if we assume any latitude for the object and substitute that angle for λ, the result will be the observed

weight of the object, correct to six decimal places, — showing that no appreciable error was caused by dropping the term $0.0035^2 m^2 \cos^2 \lambda$.

The relation between w_1 and w_0 can be expressed by still another formula. In the right triangle PBH,

$$w_1 = b \sin l + (a - c) \cos l,$$

where l is the angle that the direction of a plumb line at P would make with the equatorial radius. Substituting for a, b, and c their values, as before,

$$w_1 = w_0 \sin \lambda \sin l + (w \cos \lambda - .0035\, m \cos \lambda) \cos l, \text{ or}$$

$$w_1 = w_0 \sin \lambda \sin l + w_0 \cos \lambda \cos l - .0035\, m \cos \lambda \cos l.$$

The angles l and λ are so nearly equal that we can disregard any difference between their respective sines or cosines without producing error within the sixth decimal place. We can also replace m by w_0, as already explained. Substituting λ for l, and w_0 for m, the equation becomes

$$w_1 = w_0 \sin^2 \lambda + w_0 \cos^2 \lambda - .0035\, w_0 \cos^2 \lambda,$$

or $$w_1 = w_0 (\sin^2 \lambda + \cos^2 \lambda) - .0035\, w_0 \cos^2 \lambda,$$

or $$w_1 = w_0 - .0035\, w_0 \cos^2 \lambda,$$

or $$w_1 = w_0 (1 - .0035 \cos^2 \lambda). \qquad \text{(b)}$$

EXAMPLES

1. *On account of the flattening of the earth, a radius from the center of the earth to the mouth of the Mississippi River is longer than one drawn to the head waters of the same river, whence it appears that the river flows uphill. Why is this possible?*

2. *What will be the weight of a one-pound mass in latitude 60 degrees, corrected for loss due to the earth's rotation? Solve by equations* (a) *and* (b) *successively and compare results.*

3. *What would have been the corrected weight of the same mass in latitude of 20 degrees?*

4. *If the mean equatorial radius of the earth is 637,815,000 centimeters and its mean polar radius is 635,640,000 centimeters, what are the relative values of the gravitational attractions at the two places?*

5. *Show that the effect of the earth's rotation in diminishing weight is greatest at the equator and reduces to zero at the poles.*

The Tides. — In Fig. 152, M represents the center of the moon, and the surface of the paper represents the plane of the orbits of the earth and moon around their common center of gravity C. The larger circle may represent either an equatorial section of the earth, or a section through any parallel of latitude, — provided it is assumed that the equator

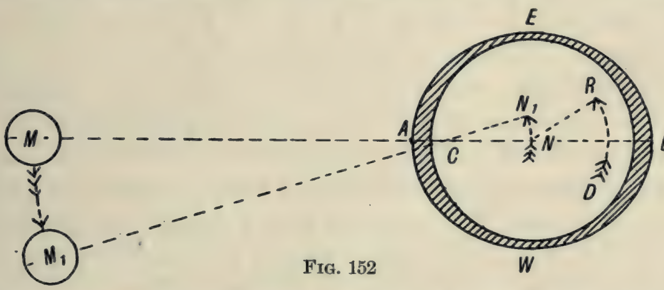


FIG. 152

is in the plane of the moon's orbit. Along every parallel of latitude the waters of the earth are heaped up toward the meridians nearest and farthest from the moon. As the rigid earth, represented by the inner unshaded circle, moves in diurnal rotation around its axis N in direction of the arrow DR, it drags the waters along with it, but the summits of the heaps tend to remain in the same positions relatively to the moon. To an observer on the earth the effect is the same as if the solid earth were at rest and a wave crest propagated through the water in the direction WAE. In the course of twenty-four hours the waters adjacent to any point on the solid earth pass in alternation and succession positions where they will be for a time heaped up and then depressed.

The forces that cause this heaping up, however, are not due to the earth's axial rotation; on the contrary, the effect of diurnal rotation in modifying weight is the same for all places of the same latitude (and altitude), as already explained. The cause of the tides lies in lunar and solar attractions, the moon exercising the greater influence because of its nearness, or rather because the radius of the earth as compared with its distance from the moon is much greater than it is in comparison with the distance from the sun. Assuming that the distance MN is 250,000 miles, then MA is 246,000 miles, and MB is 254,000 miles — provided $AWBE$ is an equatorial section. Gravitational attraction being inversely as the square of the distance, the attraction between the moon and a given mass at A will be greater than the attraction between the moon and a similar mass at N, in the ratio $(\frac{250000}{246000})^2$, or as $1.0328:1$. Likewise the attraction between the moon and a mass at N is greater than for a similar mass at B, in the ratio $(\frac{254000}{250000})^2$, or $1.0323:1$. The distance from the earth to the sun, center to center, is about 92,000,000 miles, in comparison with which the earth's radius is insignificant, but the vastly greater mass of the sun makes up much of its disadvantage of distance. Everything considered, its tide-producing influence is nearly half as great as that of the moon.

If the earth and the moon were held stationary, the water would be heaped up at A only, because in that event the weight of a given column of water at A, more than at any other point, would be diminished by the moon's attraction, and hydrostatic equilibrium would not be established until the depth of water at A became greater in proportion as its weight was diminished — just as in a U-tube a taller column of a light liquid is required to balance a shorter column of a heavier one. The fact that the water is heaped up at both A and B, and not at A alone, is because the earth is not immovably fixed, but is continuously accelerated toward the

moon by gravitational attraction. The rigid earth is attracted to the moon more strongly than the more remote water at B, and hence is like an elevator continuously starting downward, tending to leave behind the water at B, which is, therefore, under diminished hydrostatic pressure as compared with the water to the east or west. The water at A, being nearer to the moon than the center of mass of the rigid earth, tends to leave the latter behind, and hence is also under a diminished hydrostatic pressure. At both places, A and B, therefore, the influence of the moon diminishes the weight that the water would have had by virtue of the earth's attraction alone.

The moon completes one revolution in about 28 days, and hence moves from M to M_1, Fig. 152, through an angle of $\frac{360}{28}$, or 12°.5, in one day. If a point on the earth's surface starts from A and is carried through one diurnal rotation, when it returns to A the moon will have moved to M_1. Let t = the number of hours before A will overtake it. The velocity of A is $\frac{360}{24}$, or 15°, per hour. The velocity of the moon is $\frac{12.5}{24}$, or 0°.52, per hour. With a start of 12°.5, the angular distance traveled by the moon before it is overtaken will be $12.5 + .52\,t$, wherefore $12.5 + .52\,t = 15\,t$. Solving, $t = .86$ hour, or 51.6 minutes. This explains the well-known fact that the tides occur about 50 minutes later from day to day. The actual interval is nearly 54 minutes; we used in round numbers 28 days as the moon's orbital period, while the true time is 27 days and a fraction.

Tidal computations involve many factors that are not included in our simple demonstration. In Fig. 152 the plane of the earth's equator was assumed to be coincident with the plane of the moon's orbit, while the fact is that the orbital planes of the moon around the earth and of the earth around the sun make appreciable angles with each other, and with the plane of the earth's equator, all serving to complicate the calculations. Even after all astronomical factors have been taken into consideration, there are left

many local conditions that would seem to place the matter beyond the range of mathematical inquiry. Friction, irregularities of coast line, and other influences may cause a considerable difference of time in the appearance of a particular tide at two places only a short distance apart, and there are many other seemingly unaccountable circumstances. Nevertheless, for a given place the tidal phenomena repeat themselves at faithful intervals, and when once observed in relation to the moon and sun, their recurrence can be predicted for years to come, practically without limit.

EXAMPLES

1. *What are spring tides? Neap tides?*

The Conical Pendulum.— A mass m, Fig. 153, is attached to a string or rod and the latter is swiveled at A, permitting the mass to describe a horizontal circle constituting the base of a cone of which A is the apex and AO is the vertical axis. The motion described is that of a conical or "Maypole" swing, —a familiar piece of playground apparatus having a number of ropes swiveled at the top of a vertical pole, the players grasping handles at the lower ends of the ropes and, by getting up motion, swinging clear of the ground in a horizontal circle. In the figure, mOm_1 is the diameter of the horizontal circle. The greater the velocity of m in the circle, the larger the angle θ at the apex of the cone. The weight w of the mass m, being a force of given magnitude and always vertical in direction, the device must

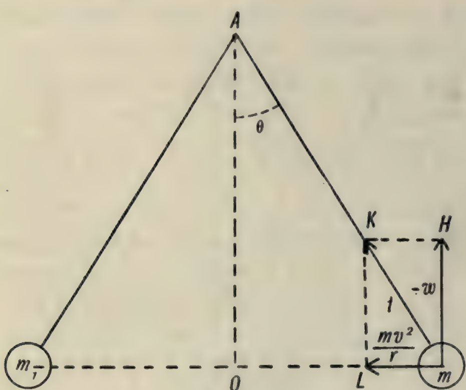


Fig. 153

adjust itself in a position where the horizontal centripetal force $\frac{mv^2}{r}$ will be equal in magnitude to HK in the triangle mHK, of which the side mH is equal to $-w$, because the tension t in the rope or rod must furnish the two components necessary to hold up the weight of the mass, and to keep it in the circle. For a given lineal velocity in the circle the formula $\frac{mv^2}{r}$ shows that the necessary centripetal force will decrease as r increases, while at the same time the horizontal component of the tension t increases ; hence when m is once started, it will move outward or inward until a condition of equilibrium is established.

If l is the length in feet of the rod mA and r is the radius mO for the position of equilibrium, then $r = l \sin \theta$. If m is in pounds, the centripetal force mL is $\frac{mv^2}{g_0 r}$ pounds, where v is in feet per second.

If w is the weight of m in pounds, then $\frac{mv^2}{g_0 r} = w \tan \theta$. But $w = \frac{mg}{g_0}$ (p. 204), whence $\tan \theta = \frac{mv^2}{g_0 r} \div \frac{mg}{g_0}$, or $\tan \theta = \frac{v^2}{gr}$. From this it appears that the angle of equilibrium is independent of the mass at a given place, and also, since the value of g varies from place to place, so will the value of θ for a given v.

The principle of the conical pendulum is used in engine governors of the Watt type, which control a valve opening in the inlet steam pipe by means of rods hinged to a pair of arms loaded and rotating like the one in Fig. 153.

EXAMPLES

1. *The formula for conical pendulums may be written in the form* $gl \frac{\sin^2 \theta}{\cos \theta} = v^2$. *If* l *has a finite value, what velocity* v *will be necessary in order that* θ *may be 90°? What will be the value of* v *for* $\theta = 0°$?

2. *If the rod of a conical pendulum is one foot long from the apex to the center of the mass at the other extremity, how many rotations per minute must it make to be in equilibrium when* $\theta = 30°$? *Assume that it is located at the equator, where* $g = 32.09$.

3. *A person grasps the handle of one of the ropes of a conical swing and gets up a velocity of 10 feet per second. If the distance from the top of the pole to the center of gravity of the person is 15.53 feet, what will be the angle θ ? Assume* g = *32.2.*

Hint: The formula may be written $\frac{\sin^2\theta}{\cos\theta} = \frac{v^2}{gl}$, or $\frac{1-\cos^2\theta}{\cos\theta} = \frac{v^2}{gl}$, or $1-\cos^2\theta = \frac{v^2\cos\theta}{gl}$, or $\cos^2\theta + \frac{v^2\cos\theta}{gl} = 1$. Substituting the given values for g, v, and l, the resulting quadratic equation is easily solved.

4. *When a person runs in circular path, at every step he not only lifts his weight, but he must also exert an extra effort to supply the centripetal force necessary to keep his motion in the circle, and not run off at a tangent. To maintain his equilibrium he inclines his body in the direction of the resultant of a vertical force* $-w$, *equal and opposite to his weight, and the horizontal central force* $mr\omega^2$ *or* $\frac{mv^2}{r}$. *He describes the frustum of a cone, the axis of which is a vertical through the center around which the circular path is laid.*

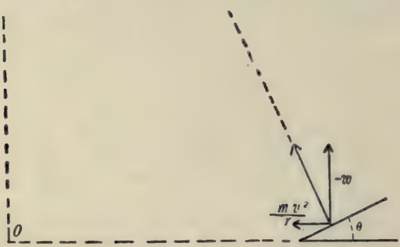


FIG. 154

If he is running at the rate of 100 yards in 12 seconds on a curve of 50-foot radius, what must be the angle of elevation of the track in order that his body may be perpendicular to it ?

5. *What would be the proper angle for a track of the same radius, if it is to be used by bicyclists at a speed of a mile in 2 minutes ?*

6. *What would be the proper angle for a railroad curve of 500-foot radius, permissible speed 30 miles an hour ?*

Centrifugal Separators. — Liquids of different densities placed in a vessel and rotated tend to separate, the heaviest liquid assuming a position against the outer walls of the ves-

sel, and the lightest innermost. Cream separators operate in this manner. In Fig. 155 a ball B is shown in a smooth tube, the latter rotating around a vertical axis O. The ball tends to continue in the tangential direction BB', and would be shot out of the tube if it were open. But, if the tube is closed and filled with a liquid heavier than the ball, the latter will move toward the axis. The crowding of the heavier liquid against the outer end of the tube floats the ball in the opposite direction, just as the weight of a liquid acting downward will buoy up objects immersed in it, according to Archimedes' principle. As globules of cream are crowded to the surface of a quantity of milk by the heavier constituents working downward under gravitational attraction, so will they be impelled toward the axis of the centrifugal machine.

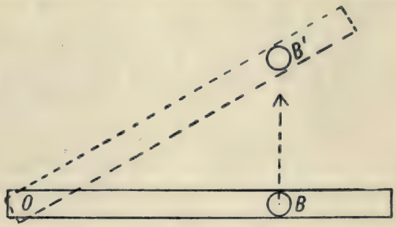


Fig. 155

All centrifugal machines are not operated on this principle of flotation. Centrifugal drying machines merely retain the solid substances placed in them and allow the liquid portion to pass between the solid particles and through openings in the outer walls of the vessel.

Elliptical Orbits; Kepler's Laws. — The earth and all the other planets move around the sun, not in true circles, but in elliptical orbits. In each instance the sun is situated at one of the foci of the ellipse, as at S in Fig. 156. In the course of its revolution, the planet is nearest the sun when it occupies the position X, and is farthest from the sun when it reaches A, at the opposite extremity of the major axis of the ellipse. The earth, for example, is nearest the sun in December, and is about 3,000,000 miles farther away in June. The distance CS is therefore about 1,500,000 miles,

and since the average distance of the earth from the sun is about 92,000,000 miles, it follows that CS for the earth's orbit is about $\frac{1}{60}$ of CX.

Exaggerating the ellipse for purposes of construction, let E be any position of the earth in its orbit. Let a represent the gravitational attraction between the earth and the sun, and let t and n be components of a, one tangential and the other normal. The component n is the deflecting or centripetal force that holds the planet in its orbit. The component t changes the orbital velocity of the planet, and hence introduces a new consideration not found in circular motion. While the body is moving from X to A its motion is retarded; from A to X it is accelerated, as shown by the direction of the tangential component at E'. Its lineal velocity is greatest at X, and least at A.

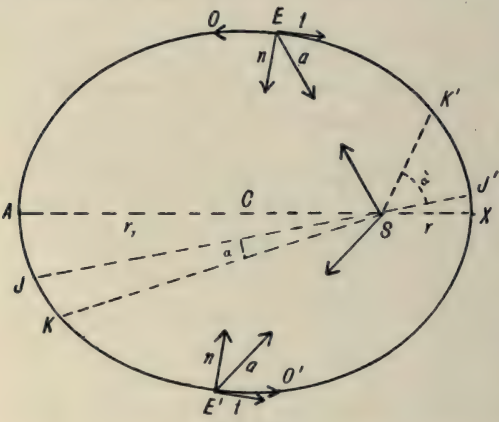


FIG. 156

The conditions are such that a line drawn from the sun to the planet — the radius vector, as it is called — sweeps through equal areas or sectors in equal time intervals, in different parts of the orbit. This is one of the laws of planetary motion worked out by Kepler in the early part of the seventeenth century from astronomical observations, and afterward used to verify Newton's law of gravitation. In the neighborhood of X the radius vector SJ' is less than SJ in the neighborhood of A, but the distance $J'K'$, traveled in a given time, will be greater than the corresponding

distance JK. A full treatment of this subject, however, requires more than elementary mathematics and cannot be undertaken here.

Kepler also observed that the different planets in their movements around the sun preserved a definite relation between their distances from the sun and the times required for completing their respective circuits. He found that the squares of their periodic times are in the same ratio as the cubes of their respective distances — and this empirical deduction, like his law of equal areas, fully confirmed the law of gravitation and the conclusions concerning central forces. Let r_1 and r_2 be the distances of any two planets from the sun, and let T_1 and T_2 be their respective periods of revolution. The lengths of their orbits will be $2\pi r_1$ and $2\pi r_2$, and the orbital velocities will be $\frac{2\pi r_1}{T_1}$ and $\frac{2\pi r_2}{T_2}$. If m_1 and m_2 are the masses of the planets, the centripetal forces will be $\frac{m_1 v_1^2}{r_1}$, or $\frac{m_1}{r_1}\left(\frac{2\pi r_1}{T_1}\right)^2$, or $\frac{4\pi^2 m_1 r_1}{T_1^2}$ for the first, and $\frac{4\pi^2 m_2 r_2}{T_2^2}$ for the second. If M is the mass of the sun, the gravitational attractions which serve to hold the planets in their orbits will be $\frac{KMm_1}{r_1^2}$ and $\frac{KMm_2}{r_2^2}$, where K is a constant depending on the units of mass and distance employed. Therefore, $\frac{KMm_1}{r_1^2}=\frac{4\pi^2 m_1 r_1}{T_1^2}$, or $\frac{r_1^3}{T_1^2}=\frac{KM}{4\pi^2}$ for the first planet, and $\frac{KMm_2}{r_2^2}=\frac{4\pi^2 m_2 r_2}{T_2^2}$, or $\frac{r_2^3}{T_2^2}=\frac{KM}{4\pi^2}$ for the second planet. Hence, $\frac{r_1^3}{T_1^2}=\frac{r_2^3}{T_2^2}$, or $\frac{r_1^3}{r_2^3}=\frac{T_1^2}{T_2^2}$.

The confirmation of these different laws in this manner was one of the greatest triumphs in the history of the inductive sciences. Kepler took the best available observations, and from them deduced and analyzed the motions of the

planets — a problem in kinematics, purely. Newton dealt with the forces controlling these motions, and it was from his work in this connection that the subject of kinetics had its real beginning. It may even be said that mechanics as a science dates from that time. Kepler's laws not only apply to the present orbits of the different planets, but they are also used to calculate the variations of each of these orbits in past and future ages.

The Hodograph. — In Chapter IV the graphical representation of acceleration was accomplished by means of two rectangular axes, as in Fig. 33, page 48. Another device for the same purpose but having a wider range of applicability is the method of the hodograph, which was introduced about a half-century ago by Sir William Hamilton. Suppose that a body is moving in the circumference of the larger circle shown in Fig. 157, with a uniform velocity represented by the tangent OX or BY. Taking any convenient point O as an origin, draw a line Ox parallel and equal to AX. If from each point in the circumference of the larger circle a tangent were drawn to represent the magnitude and direction of the orbital velocity at that point, all the lines thus drawn would be equal in length but of different directions. If from the origin O a line were drawn parallel and equal to each of the tangents of the larger circle, there would be an infinite number of such lines, like Ox and Oy, radiating from O; and if their extremities were connected, the result would be a polygon of an infinite number of sides — in other words, a circle of radius Ox. Let AX and BY be any adjacent tangents — their distance apart being exaggerated for purposes of construction. Then the line xy, one of the many sides of the polygon, will represent the change of velocity from AX to BY, because when

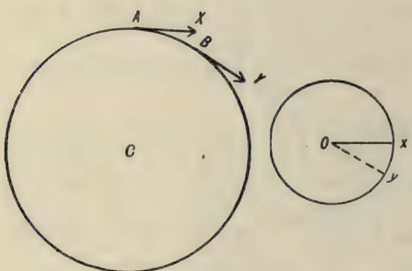


Fig. 157

xy is compounded with Ox, the resultant will be Oy. If the change from AX to BY occurred in unit time, then xy represents the acceleration. The element xy is perpendicular to AX, confirming our previous statements that a body moving in a circle with uniform velocity is constantly accelerated toward the center.

The method of the hodograph is readily applied to cases of variable acceleration —which heretofore we have expressly omitted. The curve AE, Fig. 158, being the path of the body, and the tangents AA', BB', etc., being the velocities at points A, B, C, etc., the hodograph will be the curve constructed by connecting the extremities of lines emanating from O parallel and equal to lines representing the velocities at successive points in the curve AE.

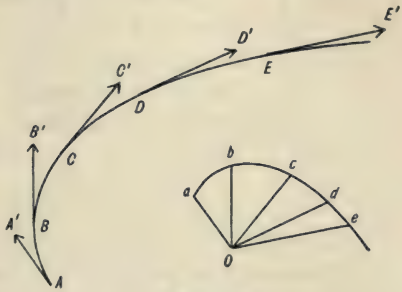


Fig. 158

For uniform acceleration in a given direction, as in the case of falling bodies, the hodograph is a straight line parallel to the direction of the velocity. Using the same conditions that were illustrated in Fig. 33, page 48, let AE, Fig. 159, be the path of the body, which was first observed to have a velocity of 10 miles an hour, represented by AB. At the end of one minute it will have traveled $\frac{11}{60}$ths of a mile, represented by AX (the scale of distances being independent of the scale of velocities), and its velocity at the end of the minute will have become 12 miles per hour, represented by XC. At the end of 2 minutes it will have reached a point Y, $\frac{13}{60}$ths of a mile from X or $\frac{24}{60}$ths from A, and its velocity will have become 14 miles an hour, represented by YD; etc. If from O as an origin we lay off lines Ob, Oc, Od, equal to 10, 12, 14, etc., the line Oe will be the hodograph of AE, and one

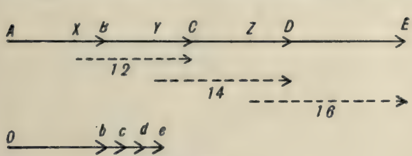


Fig. 159

of the equal intervals *bc, cd,* etc., will represent in magnitude and direction the constant acceleration.

A projectile hurled horizontally is subject to a uniform acceleration vertically downward, and this the hodograph of its path will show. In Fig. 37, page 60, the tangential velocities are indicated at several different points. Beginning with the horizontal velocity of projection, lay off lines parallel and equal to these tangential velocities, as shown in Fig. 160, using C as an origin. The line DEF is the hodograph of the parabolic path of the projectile, and the uniform acceleration is represented by each of the lines DE, EF, etc.

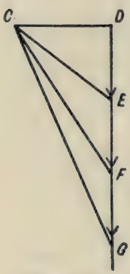


FIG. 160

EXAMPLE

Construct a curve from the conditions stated in Example 2, page 64. At successive points in the curve draw tangents, using any suitable scale, to represent the velocities in the curve, and from them construct the hodograph.

CHAPTER XIII

THE PENDULUM

The Simple Pendulum. — The ideal pendulum would have for a bob a "heavy particle," swinging by means of a "weightless fiber." An approximation for experimental purposes is a small sphere of iron or lead suspended by a very fine wire. When such a pendulum is set swinging, its rate of oscillation depends in part on its length. The shorter the pendulum the faster it swings, but not in simple proportion. A pendulum that beats seconds must be four times as long as one that beats half seconds, and nine times as long as one that beats three times a second. Or, conversely, the time required for one beat varies directly as the square root of the length, or $t \propto \sqrt{l}$. The length is measured from the point of support to the center of the bob.

The rate of oscillation for a given pendulum also changes from place to place, the time of beat varying inversely as the square root of the acceleration of gravity, or $t \propto \sqrt{\frac{1}{g}}$.

Combining these laws, $t \propto \sqrt{\frac{l}{g}}$. One beat of a pendulum is the swing from one extremity of the arc to the opposite extremity. A seconds pendulum, therefore, requires two seconds for a complete oscillation. This is called the period of oscillation, or time of a double beat. At a place where $g = 32.2$ feet per second per second, the length of a seconds pendulum is 39.15 inches.

The amplitude of vibration refers to the distance from the lowest point in the arc to one of its extremities. If the

swing is small, — not more than a few degrees, — the rate of oscillation is practically independent of the amplitude.

EXAMPLES

1. *If a 2-foot pendulum beats at a certain rate, what will be the time of beat of a 3-foot pendulum at the same place?*

2. *If a given pendulum beats at a given rate on the earth's surface, what will be its time of beat 4000 miles above the earth's surface?*

3. *If a pendulum of given length beats at a given rate at the earth's surface, what will be the length of another pendulum in order that it may beat at the same rate 4000 miles above the earth's surface?*

4. *If a given pendulum beats seconds on the earth's surface, what would be its period of oscillation if transported to the planet referred to in Example 8, page 77.*

Stated in the form of an equation, the law of the pendulum is $T = \pi\sqrt{\frac{l}{g}}$, where T is the period of oscillation in seconds, l the length of the pendulum in feet, and g the acceleration of gravity in feet per second per second. A full demonstration of this law is not usually attempted in elementary texts. When the bob B, Fig. 161, is in a position B_0 at one extremity of its arc, the component of g perpendicular to OB_0 is the only force producing rotation of the pendulum around the point of support O. The component b produces only tension in the thread. As the pendulum swings to the left the magnitude of component a decreases, becoming zero at the lowest point of the arc. The swing of the pendulum, therefore,

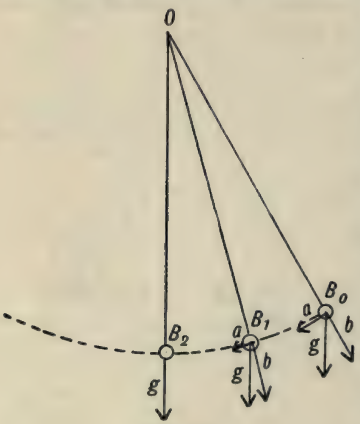


Fig. 161

is not a case of uniformly accelerated motion (beyond which we have not studied up to this time). However, the acceleration changes in a way that we can reduce to algebraic expression. Starting from the initial position B_0, Fig. 162, the tangential velocity of the bob at B_1 is equal to the vertical velocity it would acquire in falling through the distance MB_1. Let v be this velocity, and let $MB_1 = h$. Then $v = \sqrt{2\,gh}$. But $h = l - l \cos\theta = l\,(1 - \cos\theta)$, where l is the length of the pendulum in feet. Therefore, $v = \sqrt{2\,gl\,(1 - \cos\theta)}$. It is necessary now to find the time required for the pendulum to swing from B_0 to B_1. With a radius equal to MB_0, describe

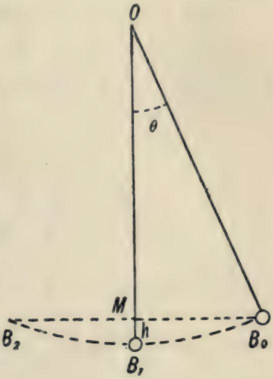


FIG. 162

a circle, as in Fig. 163. Draw diameters B_0MB_2 and AMH, perpendicular to each other. If a particle were to start at B_0, Fig. 163, and move in the circumference of the circle with a velocity equal to that of the pendulum at the lowest point of its arc, or $v = \sqrt{2\,gl\,(1 - \cos\theta)}$, it can be shown that the time necessary for this particle to cover the distance B_0A, Fig. 163, is practically the same as that required by the pendulum in moving from B_0 to B_1, Fig. 162, provided the arc of the pendulum is small. Later it will be shown that this is true,

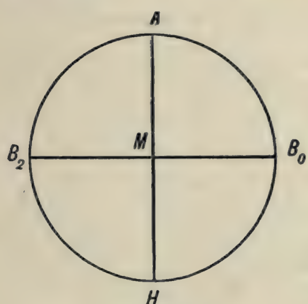


FIG. 163

because the pendulum swinging in that manner satisfies the conditions of harmonic motion. Meanwhile, from this assumption, we arrive at the periodic time of the pendulum by dividing the length of the arc B_0A, Fig. 163, by the velocity v. Since the radius $B_0M = l \sin\theta$, the arc

$B_0A = \frac{2\,\pi l \sin\theta}{4}$. Then, if t is the time from B_0 to B_1, Fig. 162, or from B_0 to A, Fig. 163,

$$t = \frac{2\,\pi l \sin\theta}{4} \div \sqrt{2\,gl\,(1-\cos\theta)}.$$

Putting $l \sin\theta$ under the radical sign,

$$t = \frac{\pi}{2}\sqrt{\frac{l^2\sin^2\theta}{2\,gl\,(1-\cos\theta)}},$$

or $$t = \frac{\pi}{2}\sqrt{\frac{l}{g}}\sqrt{\frac{\sin^2\theta}{2(1-\cos\theta)}}.$$

But $\sin^2\theta = 1 - \cos^2\theta = (1+\cos\theta)(1-\cos\theta)$,

whence $$t = \frac{\pi}{2}\sqrt{\frac{l}{g}}\sqrt{\frac{1+\cos\theta}{2}}.$$

If the angle θ is very small, $\cos\theta$ approximates unity. If $\cos\theta$ were unity, the expression $\sqrt{\frac{1+\cos\theta}{2}}$ would become $\sqrt{\frac{1+1}{2}}$, or unity. Therefore, for a very small angle θ, the time from B_0 to B_1, Fig. 162, would be approximately $t = \frac{\pi}{2}\sqrt{\frac{l}{g}}$, and if T is the time of one beat from B_1 to B_2, $T = 2\,t$, or $T = \pi\sqrt{\frac{l}{g}}$.

EXAMPLE

Find the numerical value of the expression $\sqrt{\frac{1+\cos\theta}{2}}$ *when* $\theta = 2°$ *and see whether it would make any appreciable difference if this factor were retained in the formula* $\mathrm{T} = \pi\sqrt{\frac{1}{\mathrm{g}}}\sqrt{\frac{1+\cos\theta}{2}}$.

If it makes no material difference, then we may safely discard this factor and say that for arcs smaller than 4° the rate of oscillation of a pendulum is independent of its amplitude of vibration.

Harmonic Motion. — This is a case where the motion is neither uniform nor uniformly accelerated, yet the change

of velocity occurs in such manner that the mathematical characteristics of the motion can be put in comparatively simple terms by merely extending our knowledge of uniform motion in a circle and of radial acceleration. In fact, uniform motion in a circle may be regarded as the resultant of two harmonic motions in straight lines at right angles to each other, as in HH_1 and MM_1, Fig. 164. Starting from OH as an initial position, assume that a radius sweeps around O as a center at a uniform rate. Divide the arc HM into equal parts HA, AB, etc., to represent distances passed over by the outer extremity of the radius in equal intervals of time. Let r be the length of the radius and v the lineal velocity in the circumference of the circle. From A, B, C, etc., drop perpendiculars on HO. If two particles were to start from H at the same instant, one moving in the circumference with a uniform velocity, and the other moving along the diameter HH_1 but always remaining vertically under the one moving in the circle, it is clear that the velocity of this second particle in the diameter is not uniform. It is accelerated in some manner, but not uniformly, for in that event the distance passed over would be as the square of the time $\left(d = \frac{at^2}{2}\right)$. The distance covered during six of the time intervals in Fig. 164 would be nine times as great as the distance Hb passed over in two of

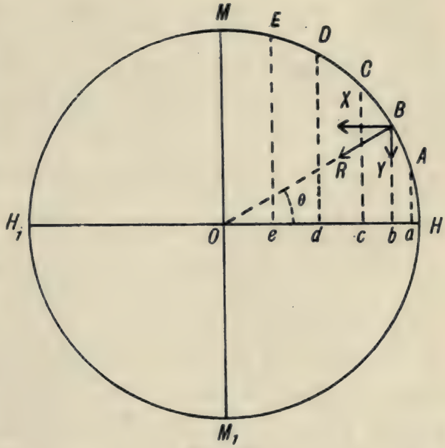


Fig. 164

the intervals, or four times as great as Hc, the distance for three intervals, etc. Measurement of the diagram will show that the proportions are otherwise.

When a body moves in a circle with uniform velocity, its radial acceleration is $\frac{v^2}{r}$. In Fig. 164 let $\frac{v^2}{r}$ be represented by the arrow BR, and let θ be the angle HOB. Resolve BR into components BX and BY parallel and perpendicular to HH_1. Then $BX = BR \cos\theta$, or $BX = \frac{v^2}{r}\cos\theta$. Since the point moving from H to O always remains in the perpendicular from the corresponding point in the circumference, it follows that this component of the radial acceleration for B, $\frac{v^2}{r}\cos\theta$, is also the acceleration for b. But $\cos\theta = \frac{Ob}{r}$, whence $\frac{v^2}{r}\cos\theta$ becomes $\frac{v^2}{r^2} \times Ob$. If we had taken the radius in position OC, the acceleration of c toward O would have been $\frac{v^2}{r^2} \times Oc$, etc. In other words, as the particle moves from H to O its acceleration is proportional to the distance from O, because v and r are both constant. In the initial position of the moving radius $\theta = 0°$ and the component $BX = BR$, or $\frac{v^2}{r}$ As θ increases, the component BX, or $\frac{v^2}{r}\cos\theta$, becomes less. When $\theta = 90°$ the particle moving in the diameter reaches O and has no acceleration.

The fact that its acceleration becomes less as the particle moves from H to O must not lead us to suppose that the velocity also diminishes. The velocity at O is greater than it was at e, and greater at e than at d, but the difference between the velocities at O and e is not as great as the difference between the velocities at e and d, or d and c. It is like the motion of the pendulum in Fig. 161, where the velocity is greatest at the lowest point of the arc, but the greater part of this velocity was acquired during the earlier part of the swing, when the descent was steepest.

The criterion of harmonic motion is that the motion is directed to a given point, and the acceleration of the moving body or particle is proportional to its distance from that point. The motion of the pendulum satisfies this test. In Fig. 165 the tangential component of the acceleration of gravity g at any point of the arc is $g \sin \theta$. But $\sin \theta = \frac{B_0M}{l}$, whence the acceleration of the pendulum at any point is $\frac{g}{l} \times B_0M$, and hence is proportional to the distance from M. For very small angles the chord tends to coincide with the arc, or point M with B_1, and it was on the idea of a very small angle that we based our assumption that the period of oscillation for the pendulum from B_0 to B_1 in Fig. 162 was the time in which a particle would move from B_0 to A in the circumference of a circle described with a radius MB_0, as in Fig. 163, having a uniform velocity equal to the velocity of the pendulum at the lowest point of its arc. The reason for using this particular velocity will now be apparent.

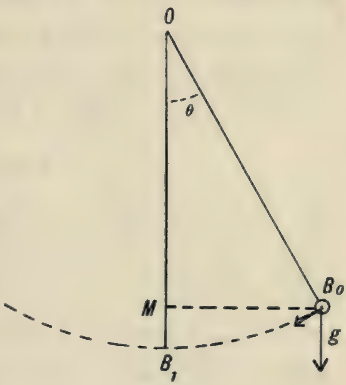


FIG. 165

In Fig. 164, when the particle moving in the diameter HH_1 reaches O, its velocity at that instant is just equal to the lineal velocity of the particle moving in the circumference. In other words, if the swing of the pendulum is so small that its arc can be assumed to be coincident with the chord, then the motion of the pendulum may be considered to be the same as that of the particle moving from H to O in Fig. 164.

It is not a part of our needs to extend the consideration of harmonic motion, which finds its widest application in the

study of wave motion in connection with Heat, Sound, Light, and other topics of radiant energy. If the radius OB, Fig. 164, continues to rotate past OM to OH_1, the horizontal acceleration will be negative. When the radius passes position OH_1, the foot of the perpendicular turns back from H_1 toward O and H, repeating in reverse direction the conditions of its motion from H to H_1. As the moving radius completes additional revolutions at a uniform rate, the foot of the perpendicular swings back and forth in harmonic motion with a periodic time determined by the revolving radius.

An excellent example of harmonic motion is that of the conical pendulum as viewed with the eye in the plane of the horizontal circle in which the bob swings. Looked at in this manner, the bob appears to swing back and forth in a straight line equal to the diameter of the circle.

EXAMPLES

1. *Using the formula* $T = \pi\sqrt{\frac{l}{g}}$, *find the rate of oscillation of a pendulum one meter long at a place where* $g = 32.2$ *feet per second per second.*

2. *Compute the length of a seconds pendulum at the same place.*

3. *Find the value of* g *at a place where a pendulum 30 inches long makes 36 complete oscillations or 72 beats a minute.*

The Compound Pendulum. — It is really of no consequence that the ideal pendulum cannot be realized, for there is a way by which any rotating mass may be used as a mathematically perfect equivalent of a simple pendulum. The ordinary clock pendulum, or any other pivoted body, may be regarded as an infinite number of ideal pendulums, all constrained to move at the same rate. Fig. 166 shows a rod free to rotate around a pivot at O. A particle near O would oscillate faster than one near the lower end of the

rod, if free to do so, but as a matter of fact both are compelled to oscillate at some intermediate rate. Between the two extremes, therefore, there is a point that oscillates in the same period that it would have if it were the bob of a simple pendulum, freed from its association with other parts of the rod.

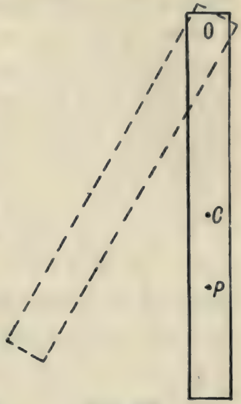


FIG. 166

For any point of suspension that may be selected in the body, as O in Fig. 166, there is somewhere on the opposite side of the center of gravity a second point P, around which the body will have the same period of oscillation.* Points O and P are thus interchangeable, and whichever is used as a center of suspension the other is called its **center of oscillation**. What is equally remarkable, the rate of oscillation of the body around either of these reciprocal points is the same as the rate of a simple or ideal pendulum of a length equal to the distance between the two points. For bodies of irregular shape the center of oscillation for a given point of suspension can be readily found by trial; for symmetrical bodies it can be computed. When the body is reversed, the former center of suspension becomes the center of oscillation.

The center of oscillation is also called the **center of percussion**, from the fact that if the object be struck at that point it will be set rotating without disturbance of the axis of rotation. An excellent illustration of this is experienced

* This is true regardless of the shape of the body. If it is a rod of uniform dimensions, it can be reversed, of course, end for end, and if pivoted at a point symmetrical with O, will rotate at the same rate. Hence, in a uniform rod, besides O and P there are two other points, symmetrical with O and P respectively. Around all of these points the rate of oscillation will be the same.

in batting a baseball. The arm and bat move around the shoulder joint as a pivot (or the bat may be swung from the wrist) and corresponding to the pivotal point there is somewhere in the bat a center of oscillation, which is the most comfortable point at which the percussion of the ball can be received. If the ball strikes swiftly at any other point of the bat, the percussion tends to wrench the shoulder (or wrist, as the case may be) and incidentally to shatter the bat.

The working out of the principle or theory of the center of oscillation was an achievement of Huygens, a contemporary of Newton. Up to that time the compound pendulum was an unsolved problem, and even Huygens' method of solution was not promptly appreciated. Its importance was in his conception of the energy of a rotating mass, — an idea that has played a prominent part in the development of mechanics.

EXAMPLE

At what rate will a body oscillate if the distance from a given point of suspension to the corresponding center of oscillation is one foot? Assume $g = 32.2$.

Determination of the Acceleration of Gravity by means of the Pendulum. — In connection with his investigations of the theory of the compound pendulum, Huygens did not fail to see its importance as an instrument of practical utility. He invented the now well-known escapement device, by means of which he made it possible to use the pendulum as an instrument for measuring time. When one of his pendulum clocks was transported to a different place it "lost time," which he explained by demonstrating that the attraction of gravity diminishes toward the equator. Since then the compound pendulum has been the standard instrument for measuring the acceleration of gravity. From the formula $T = \pi\sqrt{\frac{l}{g}}$ we get $g = \frac{\pi^2 l}{T^2}$. The experimental determination of g in this

formula requires observation of the period of oscillation for a pendulum of known length. A "pendulum of known length" can be had by taking any conveniently pivoted object, and after observing its period of oscillation, T, for the first point, reversing the object and finding by trial the other point — the center of oscillation — around which it will have the same period of oscillation. As already explained, the distance between these two points is equal to the length of an ideal pendulum having the same period, and hence gives the value of l corresponding to T in the formula.

Kater's reversible pendulum is a standard piece of apparatus for this purpose. It is a rod with knife-edges at a fixed distance apart and with adjustable weights free to slide on the rod between the knife-edges. The weights are adjusted until the apparatus is found to have the same period of oscillation when shifted from one knife-edge to the other.

EXAMPLE

The length of a clock pendulum tends to change with the temperature. A compensating pendulum is one that is designed to preserve a constant length automatically. What devices do you know of or can you think of that will accomplish this purpose?

Foucault's Pendulum. — A rifle ball fired in any horizontal direction is deflected to the right in the northern hemisphere, and to the left in the southern hemisphere, by the earth's rotation. This tendency is *nil* at the equator and increases toward the poles. It is not that the path of the bullet is curved, but the target is shifted by the earth's rotation during the flight of the bullet. Oceanic and atmospheric currents are deflected in the same way, which accounts for the clockwise circulation of the Gulf Stream and the Japan Current in the northern hemisphere and for the counter clockwise currents in the southern hemisphere.

The first physical demonstration of the earth's rotation was an experiment designed to keep a mass moving in one plane for a

period of time long enough to make this deflection visible. Imagine a long pendulum situated at the north pole. According to the principle of inertia, it would swing back and forth in one vertical plane while the earth turned beneath. To an observer moving with the earth, the pendulum would appear to be working its way around to the right, making a complete circuit in 24 hours. In 1851 Foucault suspended from the dome of the Pantheon in Paris an iron ball weighing several hundred pounds, by means of a wire more than 200 feet long, and in the course of 32 hours its plane of oscillation appeared to make a complete revolution relatively to the building in which it was situated. If the pendulum had been at the north pole, the earth would have turned under it full 360° in 24 hours. In any other latitude the daily deflection would be 360° × sine of the latitude. In Fig. 167 let E be the center of the earth, P the north pole, and V the vertex of a cone formed by drawing in the planes of the successive meridians tangents to the earth at all points on the circumference of a circle representing the "parallel of latitude" corresponding to the angle λ in the diagram. The line AB may be regarded as this circumference projected on its own diameter. Imagine a pendulum at C swinging north and south in a plane perpendicular to the plane of the paper, while the earth revolves from left to right around its axis PS. To an observer on the earth the line CV will be a north-and-south line in the plane of the horizon. As the location of the pendulum moves from C to D, the line CV assumes the position DV, and if the plane in which the pendulum is oscillating remains parallel to its first position, it will appear to have rotated to the right relatively to the earth through an angle equal to VDD_1 or CVD. In 24 hours the line CV will describe the surface of the cone, and relatively to the plane of the

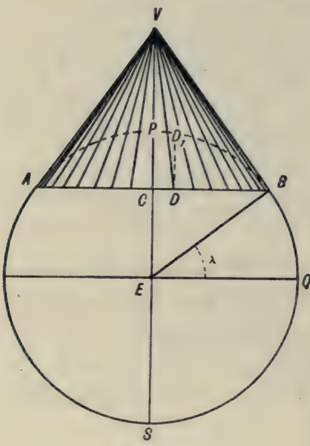


Fig. 167

pendulum will have moved to the left through an angle equal to the sum of all the small angles CVD at the vertex of the cone, or $360° \times \sin \lambda$, as may be shown. If r is the radius of the earth, then $r \cos \lambda$ is the radius of the circle at the base of the cone and $2\pi r \cos \lambda$ is its circumference. The length of one of its elements, as VB, is r cotan λ. With a radius equal to one of these elements describe a circle, Fig. 168. The circumference of this circle is $2\pi r$ cotan λ. If the cone were placed with one of its elements resting on VL and then rolled in the direction LMN with its vertex remaining at V, the circumference of its base, $2\pi r \cos \lambda$, would measure off the arc LMN. But the angle θ subtended by this arc is to 360° as the arc LMN is to the entire circumference, or $\frac{\theta}{360°} = \frac{2\pi r \cos \lambda}{2\pi r \text{ cotan } \lambda}$, whence $\theta = 360° \times \sin \lambda$.

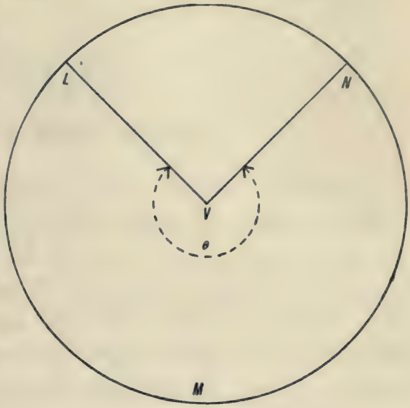


Fig. 168

CHAPTER XIV

IMPACT AND ELASTICITY

Collision. — When two masses come together there is a readjustment of their velocities and momenta, but the dynamical considerations in this instance cannot be fully explained through the laws of motion as applied in Chapter XI. During the impact the masses give up their striking velocities through mutual opposition and acquire new velocities, the whole transaction taking place in an immeasurably small interval of time. One of the laws of kinetics (p. 193) asserts that the "force-time equals change of momentum," but this avails us nothing in the case of collision, because no way has been found to measure the length of time the masses remain in contact or the stresses existing during that time. However, two controlling principles, by which the velocities after impact can be determined, were arrived at by Newton after an elaborate series of experiments, and while these do not reveal the inner secrets of the phenomenon, they afford us a good working knowledge of the subject.

First: The principle of the conservation of momentum asserts that the total momentum of the two masses after collision is the same as before; the momentum gained by one is equal to that lost by the other. *Second:* If the masses rebound, their relative velocity before collision is reversed in direction by the impact, and is diminished in a ratio that is constant for any given pair of substances.

Impact of Inelastic Masses. — If there is no rebound, the bodies are said to be devoid of elasticity. There is probably

no substance that satisfies this test perfectly, even such a material as soft putty showing some tendency to regain its form. After collision two inelastic masses move conjointly with a velocity which can be determined by the principle of the conservation of momentum, without the aid of the second principle. Let m and m_1 be the two masses, and u and u_1 their respective velocities before impact. Let v be the velocity after impact. Then $(m + m_1)v = mu + m_1u_1$;

whence $$v = \frac{mu + m_1u_1}{m + m_1}.$$

If the masses were moving in the same direction, one must have overtaken the other, so that the total momentum after impact is the sum of the two momenta before impact, as the signs of the terms in the equation indicate. If the bodies were moving in opposite directions, one of the velocities u or u_1 must be negative, and v will have the sign of the numerator of the equation, since its denominator is always positive. In other words, the principle of the conservation of momentum implies an algebraic sum, and if the two velocities are in opposite directions, one must be negative.

Loss of Energy in Impact.— Collision is accompanied by the production of sound and heat, perhaps electricity, and sometimes light. This, of course, occurs at the expense of the kinetic energy of the masses, but it does not violate the law of the conservation of momentum, as might appear at first glance. The total momentum remains the same, although the kinetic energy of the two masses is diminished, as may be shown. The kinetic energy of a moving mass is $\frac{mv^2}{2g}$, if expressed in foot-pounds, or $\frac{mv^2}{2}$ in foot-poundals, where m is the mass in pounds and v the velocity in feet per second. In the preceding paragraph the total kinetic energy of the two bodies before impact was $\frac{mu^2}{2} + \frac{m_1u_1^2}{2}$.

After impact it was $\frac{(m+m_1)v^2}{2}$, or $\frac{m+m_1}{2}\left(\frac{mu+m_1u_1}{m+m_1}\right)^2$, or $\frac{(mu+m_1u_1)^2}{2(m+m_1)}$. To show that there is a loss of energy, we must prove the energy before impact greater than afterward, or

$$\frac{mu^2}{2}+\frac{m_1u_1^2}{2}>\frac{(mu+m_1u_1)^2}{2(m+m_1)}.$$

Reducing, $$mu^2+m_1u_1^2>\frac{m^2u^2+2\,mm_1uu_1+m_1^2u_1^2}{m+m_1}.$$

Clearing of fractions, etc.,

$$m^2u^2+mm_1u_1^2+mm_1u^2+m_1^2u_1^2>m^2u^2+2\,mm_1uu_1+m_1^2u_1^2;$$

$$mm_1u_1^2+mm_1u^2>2\,mm_1uu_1,$$

$$u_1^2+u^2>2\,uu_1,$$

$$u_1^2-2\,uu_1+u^2>0,$$

$$(u_1-u)^2>0.$$

In the last expression, whether u_1-u is positive or negative, its square is positive, and hence greater than zero. This proves that there is a loss of energy. The magnitude of this loss is

$$\frac{1}{2}(mu^2+m_1u_1^2)-\frac{1}{2}\frac{(mu+m_1u_1)^2}{m+m_1},$$

or, $$\frac{1}{2}\left(\frac{mm_1u^2+mm_1u_1^2-2\,mm_1uu_1}{m+m_1}\right),\text{ or }\frac{mm_1(u-u_1)^2}{2(m+m_1)}.$$

Impact of Elastic Masses. — Two masses that rebound after collision are said to be elastic. Just as there is no substance entirely devoid of elasticity, so there is none perfectly elastic. Therefore, the relative velocity with which the masses come together is not fully restored in the rebound. If the velocity of separation is a certain fraction of the striking velocity, this fraction is called the coefficient of restitution for the given pair of substances. For example, if two masses have velocities of 30 and 40 feet per second, respectively, in the same

direction, the former leading, the second mass will overtake the first and their relative velocity will be 10 feet per second. After collision the velocity of the second will be found to have been retarded and that of the first will have been increased. If the coefficient of restitution is $\frac{82}{100}$, their relative velocity after impact will be -8.2 feet per second, or 8.2 feet per second in favor of the first mass, the relative velocity of the two having been reversed in direction as well as diminished in numerical value. In general terms, if u and u_1 are the velocities before impact, v and v_1 after impact, and ρ the coefficient of restitution, then

$$v_1 - v = -\rho\,(u_1 - u)$$

is the formula for the *relative* velocity.

To find the *actual* velocity we make use of the momentum principle. The total momentum is the same before and after impact, or

$$mu + m_1u_1 = mv + m_1v_1.$$

By combining these equations, eliminating v_1, we get

$$-\rho(u_1 - u) + v = \frac{mu + m_1u_1 - mv}{m_1},$$

or $$-\rho m_1u_1 + \rho m_1u + m_1v = mu + m_1u_1 - mv,$$

or $$mv + m_1v = mu - \rho m_1u + m_1u_1 + \rho m_1u_1,$$

or $$v = \frac{u\,(m - \rho m_1) + m_1u_1\,(1 + \rho)}{m + m_1}.$$

By eliminating v in the same manner, we get

$$v_1 = \frac{u_1\,(m_1 - \rho m) + mu\,(1 + \rho)}{m + m_1}.$$

Coefficient of Elasticity. — The coefficient of restitution is a dynamical measure of the elasticity of substances, and is used only in dealing with the phenomenon of collision. In all other parts of physics a statical measure, called the coefficient of elasticity, is used. When a body is deformed by a gradually increasing force,—as in bending a beam,—the

deformation increases in direct proportion to the deforming force. This has already been referred to (page 97) as Hooke's Law. When the body is relieved of this force, it resumes its original form. This law is obeyed up to a certain limit, called the elastic limit of the body. If the body is loaded beyond its elastic limit, it takes a permanent set, and when relieved will only partly return to its first form. Furthermore, beyond the elastic limit Hooke's Law ceases to apply, the deformation beyond that point increasing usually faster than the applied force.

The coefficient of elasticity for a given substance, as ordinarily expressed, is the force necessary to produce a unit deformation in a piece of the material of unit dimensions. For example, if a force of F pounds produces an elongation of e inches in a wire l inches long and a square inches in cross section, by analysis we can find the force necessary to produce an elongation of 1 inch in a piece 1 inch long and 1 square inch in cross section. This latter force would be the coefficient of elasticity E for the given substance, and it will be seen that

$$E = \frac{Fl}{ea}.$$

The coefficient of restitution is an abstract number and is always less than unity, while the coefficient of elasticity is a very large number — usually in the millions, if expressed in pounds per square inch.

Another statical measure of elasticity would be the deformation produced by a unit force acting on an object of unit dimensions. This is always a very small fraction and is not as convenient as the coefficient of elasticity described above.

EXAMPLES

1. *A mass of 12 pounds moving with a velocity of 10 feet per second overtakes a mass of 25 pounds moving with a velocity of 6 feet per second. If both are entirely inelastic, what will be their common velocity after impact ?*

2. *What loss of kinetic energy do they suffer from the impact?*

3. *If the same masses had been moving at the rates mentioned in the first example, but in directions opposite to each other, what would have been the velocity after impact, and what would have been the loss of energy?*

4. *A mass of 12 pounds moving with a velocity of 10 feet per second overtakes a mass of 25 pounds moving with a velocity of 6 feet per second. If the coefficient of restitution for the two masses is 0.75, find:*

(*a*) *The relative velocity before impact.*
(*b*) *The relative velocity after impact.*
(*c*) *The actual velocity of each after impact.*

5. *If the masses referred to in Example 4 had been moving in opposite directions, what would have been the answers to* (*a*), (*b*), *and* (*c*)?

6. *If the coefficient of elasticity for spring brass is 12,000,000 pounds per square inch, what will be the elongation of a wire* $\frac{1}{8}$ *inch in diameter and 10 feet long under a load of 200 pounds?*

7. *A load of 50 pounds applied to a piece of piano wire 8 feet long and* $\frac{15}{1000}$ *inch in diameter produces an elongation of* $\frac{1}{10}$ *inch. Find the coefficient of elasticity of piano wire.*

REFERENCE TABLE OF EQUATIONS

Laws of mechanics that are capable of expression in the form of algebraic equations will be found on the pages indicated in the following list, which also shows the subject to which each law pertains:

PHILLIPS AND STRONG'S

FOUR-PLACE
NATURAL TRIGONOMETRIC
FUNCTIONS

TO EVERY TEN MINUTES

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
0 0	0.0000	29	0.0000	29	infinit.	—	1.0000	0	0 90
10	0.0029	29	0.0029	29	343.7737	—	1.0000	0	50
20	0.0058	29	0.0058	29	171.8854	—	1.0000	0	40
30	0.0087	29	0.0087	29	114.5887	—	1.0000	1	30
40	0.0116	29	0.0116	29	85.9398	—	0.9999	0	20
50	0.0145	30	0.0145	30	68.7501	114601	0.9999	1	10
1 0	0.0175	29	0.0175	29	57.2900	81861	0.9998	0	0 89
10	0.0204	29	0.0204	29	49.1039	61398	0.9998	1	50
20	0.0233	29	0.0233	29	42.9641	47756	0.9997	0	40
30	0.0262	29	0.0262	29	38.1885	38207	0.9997	1	30
40	0.0291	29	0.0291	29	34.3678	31262	0.9996	1	20
50	0.0320	29	0.0320	29	31.2416	26053	0.9995	1	10
2 0	0.0349	29	0.0349	29	28.6363	22047	0.9994	1	0 88
10	0.0378	29	0.0378	29	26.4316	18898	0.9993	1	50
20	0.0407	29	0.0407	30	24.5418	16380	0.9992	2	40
30	0.0436	29	0.0437	29	22.9038	14334	0.9990	1	30
40	0.0465	29	0.0466	29	21.4704	12648	0.9989	1	20
50	0.0494	29	0.0495	29	20.2056	11245	0.9988	2	10
3 0	0.0523	29	0.0524	29	19.0811	10061	0.9986	1	0 87
10	0.0552	29	0.0553	29	18.0750	9057	0.9985	2	50
20	0.0581	29	0.0582	30	17.1693	8194	0.9983	2	40
30	0.0610	30	0.0612	29	16.3499	7451	0.9981	1	30
40	0.0640	29	0.0641	29	15.6048	6804	0.9980	2	20
50	0.0669	29	0.0670	29	14.9244	6237	0.9978	2	10
4 0	0.0698	29	0.0699	30	14.3007	5740	0.9976	2	0 86
10	0.0727	29	0.0729	29	13.7267	5298	0.9974	3	50
20	0.0756	29	0.0758	29	13.1969	4907	0.9971	2	40
30	0.0785	29	0.0787	29	12.7062	4557	0.9969	2	30
40	0.0814	29	0.0816	30	12.2505	4243	0.9967	3	20
50	0.0843	29	0.0846	29	11.8262	3961	0.9964	2	10
5 0	0.0872		0.0875		11.4301		0.9962		0 85
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	26053	16380	11245		8194	6237	4907		3961	30	29
.1	2605	1638	1125	.1	819.4	623.7	490.7	.1	396.1	3.0	2.9
.2	5211	3276	2249	.2	1638.8	1247.4	981.4	.2	792.2	6.0	5.8
.3	7816	4914	3374	.3	2458.2	1871.1	1472.1	.3	1188.3	9.0	8.7
.4	10421	6552	4498	.4	3277.6	2494.8	1962.8	.4	1584.4	12.0	11.6
.5	13027	8190	5623	.5	4097.0	3118.5	2453.5	.5	1980.5	15.0	14.5
.6	15632	9828	6747	.6	4916.4	3742.2	2944.2	.6	2376.6	18.0	17.4
.7	18237	11466	7872	.7	5735.8	4365.9	3434.9	.7	2772.7	21.0	20.3
.8	20842	13104	8996	.8	6555.2	4989.6	3925.6	.8	3168.8	24.0	23.2
.9	23448	14742	10121	.9	7374.6	5613.3	4416.3	.9	3564.9	27.0	26.1

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
5 0	0.0872	29	0.0875	29	11.4301	3707	0.9962	3	0 **85**
10	0.0901	28	0.0904	30	11.0594	3475	0.9959	2	50
20	0.0929	29	0.0934	29	10.7119	3265	0.9957	3	40
30	0.0958	29	0.0963	29	10.3854	3074	0.9954	3	30
40	0.0987	29	0.0992	30	10.0780	2898	0.9951	3	20
50	0.1016	29	0.1022	29	9.7882	2738	0.9948	3	10
6 0	0.1045	29	0.1051	29	9.5144	2591	0.9945	3	0 **84**
10	0.1074	29	0.1080	30	9.2553	2455	0.9942	3	50
20	0.1103	29	0.1110	29	9.0098	2329	0.9939	3	40
30	0.1132	29	0.1139	30	8.7769	2214	0.9936	4	30
40	0.1161	29	0.1169	29	8.5555	2105	0.9932	3	20
50	0.1190	29	0.1198	30	8.3450	2007	0.9929	4	10
7 0	0.1219	29	0.1228	29	8.1443	1913	0.9925	3	0 **83**
10	0.1248	28	0.1257	30	7.9530	1826	0.9922	4	50
20	0.1276	29	0.1287	30	7.7704	1746	0.9918	4	40
30	0.1305	29	0.1317	29	7.5958	1671	0.9914	3	30
40	0.1334	29	0.1346	30	7.4287	1600	0.9911	4	20
50	0.1363	29	0.1376	29	7.2687	1533	0.9907	4	10
8 0	0.1392	29	0.1405	30	7.1154	1472	0.9903	4	0 **82**
10	0.1421	28	0.1435	30	6.9682	1413	0.9899	5	50
20	0.1449	29	0.1465	30	6.8269	1357	0.9894	4	40
30	0.1478	29	0.1495	29	6.6912	1306	0.9890	4	30
40	0.1507	29	0.1524	30	6.5606	1258	0.9886	5	20
50	0.1536	28	0.1554	30	6.4348	1210	0.9881	4	10
9 0	0.1564	29	0.1584	30	6.3138	1168	0.9877	5	0 **81**
10	0.1593	29	0.1614	30	6.1970	1126	0.9872	4	50
20	0.1622	28	0.1644	29	6.0844	1086	0.9868	5	40
30	0.1650	29	0.1673	30	5.9758	1050	0.9863	5	30
40	0.1679	29	0.1703	30	5.8708	1014	0.9858	5	20
50	0.1708	28	0.1733	30	5.7694	981	0.9853	5	10
10 0	0.1736		0.1763		5.6713		0.9848		0 **80**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	2738	1533	981		30	29	28		5	4	3
.1	273.8	153.3	98.1	.1	3.0	2.9	2.8	.1	0.5	0.4	0.3
.2	547.6	306.6	196.2	.2	6.0	5.8	5.6	.2	1.0	0.8	0.6
.3	821.4	459.9	294.3	.3	9.0	8.7	8.4	.3	1.5	1.2	0.9
.4	1095.2	613.2	392.4	.4	12.0	11.6	11.2	.4	2.0	1.6	1.2
.5	1369.0	766.5	490.5	.5	15.0	14.5	14.0	.5	2.5	2.0	1.5
.6	1642.8	919.8	588.6	.6	18.0	17.4	16.8	.6	3.0	2.4	1.8
.7	1916.6	1073.1	686.7	.7	21.0	20.3	19.6	.7	3.5	2.8	2.1
.8	2190.4	1226.4	784.8	.8	24.0	23.2	22.4	.8	4.0	3.2	2.4
.9	2464.2	1379.7	882.9	.9	27.0	26.1	25.2	.9	4.5	3.6	2.7

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
10 0	0.1736	29	0.1763	30	5.6713	949	0.9848	5	0 **80**
10	0.1765	29	0.1793	30	5.5764	919	0.9843	5	50
20	0.1794	28	0.1823	30	5.4845	890	0.9838	5	40
30	0.1822	29	0.1853	30	5.3955	862	0.9833	6	30
40	0.1851	29	0.1883	31	5.3093	836	0.9827	5	20
50	0.1880	28	0.1914	30	5.2257	811	0.9822	6	10
11 0	0.1908	29	0.1944	30	5.1446	788	0.9816	5	0 **79**
10	0.1937	28	0.1974	30	5.0658	764	0.9811	6	50
20	0.1965	29	0.2004	31	4.9894	742	0.9805	6	40
30	0.1994	28	0.2035	30	4.9152	722	0.9799	6	30
40	0.2022	29	0.2065	30	4.8430	701	0.9793	6	20
50	0.2051	28	0.2095	31	4.7729	683	0.9787	6	10
12 0	0.2079	29	0.2126	30	4.7046	664	0.9781	6	0 **78**
10	0.2108	28	0.2156	30	4.6382	646	0.9775	6	50
20	0.2136	28	0.2186	31	4.5736	629	0.9769	6	40
30	0.2164	29	0.2217	30	4.5107	613	0.9763	6	30
40	0.2193	28	0.2247	31	4.4494	597	0.9757	7	20
50	0.2221	29	0.2278	31	4.3897	582	0.9750	6	10
13 0	0.2250	28	0.2309	30	4.3315	568	0.9744	7	0 **77**
10	0.2278	28	0.2339	31	4.2747	554	0.9737	7	50
20	0.2306	28	0.2370	31	4.2193	540	0.9730	6	40
30	0.2334	29	0.2401	31	4.1653	527	0.9724	7	30
40	0.2363	28	0.2432	30	4.1126	515	0.9717	7	20
50	0.2391	28	0.2462	31	4.0611	503	0.9710	7	10
14 0	0.2419	28	0.2493	31	4.0108	491	0.9703	7	0 **76**
10	0.2447	29	0.2524	31	3.9617	481	0.9696	7	50
20	0.2476	28	0.2555	31	3.9136	469	0.9689	8	40
30	0.2504	28	0.2586	31	3.8667	459	0.9681	7	30
40	0.2532	28	0.2617	31	3.8208	448	0.9674	7	20
50	0.2560	28	0.2648	31	3.7760	439	0.9667	8	10
15 0	0.2588		0.2679		3.7321		0.9659		0 **75**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	742	448	31		30	29	28		7	6	5
.1	74.2	44.8	3.1	.1	3.0	2.9	2.8	.1	0.7	0.6	0.5
.2	148.4	89.6	6.2	.2	6.0	5.8	5.6	.2	1.4	1.2	1.0
.3	222.6	134.4	9.3	.3	9.0	8.7	8.4	.3	2.1	1.8	1.5
.4	296.8	179.2	12.4	.4	12.0	11.6	11.2	.4	2.8	2.4	2.0
.5	371.0	224.0	15.5	.5	15.0	14.5	14.0	.5	3.5	3.0	2.5
.6	445.2	268.8	18.6	.6	18.0	17.4	16.8	.6	4.2	3.6	3.0
.7	519.4	313.6	21.7	.7	21.0	20.3	19.6	.7	4.9	4.2	3.5
.8	593.6	358.4	24.8	.8	24.0	23.2	22.4	.8	5.6	4.8	4.0
.9	667.8	403.2	27.9	.9	27.0	26.1	25.2	.9	6.3	5.4	4.5

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
15 0	0.2588	28	0.2679	32	3.7321	430	0.9659	7	0 **75**
10	0.2616	28	0.2711	31	3.6891	421	0.9652	8	50
20	0.2644	28	0.2742	31	3.6470	411	0.9644	8	40
30	0.2672	28	0.2773	32	3.6059	403	0.9636	8	30
40	0.2700	28	0.2805	31	3.5656	395	0.9628	7	20
50	0.2728	28	0.2836	31	3.5261	387	0.9621	8	10
16 0	0.2756	28	0.2867	32	3.4874	379	0.9613	8	0 **74**
10	0.2784	28	0.2899	32	3.4495	371	0.9605	9	50
20	0.2812	28	0.2931	31	3.4124	365	0.9596	8	40
30	0.2840	28	0.2962	32	3.3759	357	0.9588	8	30
40	0.2868	28	0.2994	32	3.3402	350	0.9580	8	20
50	0.2896	28	0.3026	31	3.3052	343	0.9572	9	10
17 0	0.2924	28	0.3057	32	3.2709	338	0.9563	8	0 **73**
10	0.2952	27	0.3089	32	3.2371	330	0.9555	9	50
20	0.2979	28	0.3121	32	3.2041	325	0.9546	9	40
30	0.3007	28	0.3153	32	3.1716	319	0.9537	9	30
40	0.3035	27	0.3185	32	3.1397	313	0.9528	8	20
50	0.3062	28	0.3217	32	3.1084	307	0.9520	9	10
18 0	0.3090	28	0.3249	32	3.0777	302	0.9511	9	0 **72**
10	0.3118	27	0.3281	33	3.0475	297	0.9502	10	50
20	0.3145	28	0.3314	32	3.0178	291	0.9492	9	40
30	0.3173	28	0.3346	32	2.9887	287	0.9483	9	30
40	0.3201	27	0.3378	33	2.9600	281	0.9474	9	20
50	0.3228	28	0.3411	32	2.9319	277	0.9465	10	10
19 0	0.3256	27	0.3443	33	2.9042	272	0.9455	9	0 **71**
10	0.3283	28	0.3476	32	2.8770	268	0.9446	10	50
20	0.3311	27	0.3508	33	2.8502	263	0.9436	10	40
30	0.3338	27	0.3541	33	2.8239	259	0.9426	9	30
40	0.3365	28	0.3574	33	2.7980	255	0.9417	10	20
50	0.3393	27	0.3607	33	2.7725	250	0.9407	10	10
20 0	0.3420		0.3640		2.7475		0.9397		0 **70**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	255	33	32		31	28	27		10	9	8
.1	25.5	3.3	3.2	.1	3.1	2.8	2.7	.1	1.0	0.9	0.8
.2	51.0	6.6	6.4	.2	6.2	5.6	5.4	.2	2.0	1.8	1.6
.3	76.5	9.9	9.6	.3	9.3	8.4	8.1	.3	3.0	2.7	2.4
.4	102.0	13.2	12.8	.4	12.4	11.2	10.8	.4	4.0	3.6	3.2
.5	127.5	16.5	16.0	.5	15.5	14.0	13.5	.5	5.0	4.5	4.0
.6	153.0	19.8	19.2	.6	18.6	16.8	16.2	.6	6.0	5.4	4.8
.7	178.5	23.1	22.4	.7	21.7	19.6	18.9	.7	7.0	6.3	5.6
.8	204.0	26.4	25.6	.8	24.8	22.4	21.6	.8	8.0	7.2	6.4
.9	229.5	29.7	28.8	.9	27.9	25.2	24.3	.9	9.0	8.1	7.2

°	′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	′	°
20	0	0.3420	28	0.3640	33	2.7475	247	0.9397	10	0	70
	10	0.3448	27	0.3673	33	2.7228	243	0.9387	10	50	
	20	0.3475	27	0.3706	33	2.6985	239	0.9377	10	40	
	30	0.3502	27	0.3739	33	2.6746	235	0.9367	11	30	
	40	0.3529	28	0.3772	33	2.6511	232	0.9356	10	20	
	50	0.3557	27	0.3805	34	2.6279	228	0.9346	10	10	
21	0	0.3584	27	0.3839	33	2.6051	225	0.9336	11	0	69
	10	0.3611	27	0.3872	34	2.5826	221	0.9325	10	50	
	20	0.3638	27	0.3906	33	2.5605	219	0.9315	11	40	
	30	0.3665	27	0.3939	34	2.5386	214	0.9304	11	30	
	40	0.3692	27	0.3973	33	2.5172	212	0.9293	10	20	
	50	0.3719	27	0.4006	34	2.4960	209	0.9283	11	10	
22	0	0.3746	27	0.4040	34	2.4751	206	0.9272	11	0	68
	10	0.3773	27	0.4074	34	2.4545	203	0.9261	11	50	
	20	0.3800	27	0.4108	34	2.4342	200	0.9250	11	40	
	30	0.3827	27	0.4142	34	2.4142	197	0.9239	11	30	
	40	0.3854	27	0.4176	34	2.3945	195	0.9228	12	20	
	50	0.3881	26	0.4210	35	2.3750	191	0.9216	11	10	
23	0	0.3907	27	0.4245	34	2.3559	190	0.9205	11	0	67
	10	0.3934	27	0.4279	35	2.3369	186	0.9194	12	50	
	20	0.3961	26	0.4314	34	2.3183	185	0.9182	11	40	
	30	0.3987	27	0.4348	35	2.2998	181	0.9171	12	30	
	40	0.4014	27	0.4383	34	2.2817	180	0.9159	12	20	
	50	0.4041	26	0.4417	35	2.2637	177	0.9147	12	10	
24	0	0.4067	27	0.4452	35	2.2460	174	0.9135	11	0	66
	10	0.4094	26	0.4487	35	2.2286	173	0.9124	12	50	
	20	0.4120	27	0.4522	35	2.2113	170	0.9112	12	40	
	30	0.4147	26	0.4557	35	2.1943	168	0.9100	12	30	
	40	0.4173	27	0.4592	36	2.1775	166	0.9088	13	20	
	50	0.4200	26	0.4628	35	2.1609	164	0.9075	12	10	
25	0	0.4226		0.4663		2.1445		0.9063		0	65
		Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′	°

PP	177	35	34		33	27	26		12	11	10
.1	17.7	3.5	3.4	.1	3.3	2.7	2.6	.1	1.2	1.1	1.0
.2	35.4	7.0	6.8	.2	6.6	5.4	5.2	.2	2.4	2.2	2.0
.3	53.1	10.5	10.2	.3	9.9	8.1	7.8	.3	3.6	3.3	3.0
.4	70.8	14.0	13.6	.4	13.2	10.8	10.4	.4	4.8	4.4	4.0
.5	88.5	17.5	17.0	.5	16.5	13.5	13.0	.5	6.0	5.5	5.0
.6	106.2	21.0	20.4	.6	19.8	16.2	15.6	.6	7.2	6.6	6.0
.7	123.9	24.5	23.8	.7	23.1	18.9	18.2	.7	8.4	7.7	7.0
.8	141.6	28.0	27.2	.8	26.4	21.6	20.8	.8	9.6	8.8	8.0
.9	159.3	31.5	30.6	.9	29.7	24.3	23.4	.9	10.8	9.9	9.0

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
25 0	0.4226	27	0.4663	36	2.1445	162	0.9063	12	0 65
10	0.4253	26	0.4699	35	2.1283	160	0.9051	13	50
20	0.4279	26	0.4734	36	2.1123	158	0.9038	12	40
30	0.4305	26	0.4770	36	2.0965	156	0.9026	13	30
40	0.4331	27	0.4806	35	2.0809	154	0.9013	12	20
50	0.4358	26	0.4841	36	2.0655	152	0.9001	13	10
26 0	0.4384	26	0.4877	36	2.0503	150	0.8988	13	0 64
10	0.4410	26	0.4913	37	2.0353	149	0.8975	13	50
20	0.4436	26	0.4950	36	2.0204	147	0.8962	13	40
30	0.4462	26	0.4986	36	2.0057	145	0.8949	13	30
40	0.4488	26	0.5022	37	1.9912	144	0.8936	13	20
50	0.4514	26	0.5059	36	1.9768	142	0.8923	13	10
27 0	0.4540	26	0.5095	37	1.9626	140	0.8910	13	0 63
10	0.4566	26	0.5132	37	1.9486	139	0.8897	13	50
20	0.4592	25	0.5169	37	1.9347	137	0.8884	14	40
30	0.4617	26	0.5206	37	1.9210	136	0.8870	13	30
40	0.4643	26	0.5243	37	1.9074	134	0.8857	14	20
50	0.4669	26	0.5280	37	1.8940	133	0.8843	14	10
28 0	0.4695	25	0.5317	37	1.8807	131	0.8829	13	0 62
10	0.4720	26	0.5354	38	1.8676	130	0.8816	14	50
20	0.4746	26	0.5392	38	1.8546	128	0.8802	14	40
30	0.4772	25	0.5430	37	1.8418	127	0.8788	14	30
40	0.4797	26	0.5467	38	1.8291	126	0.8774	14	20
50	0.4823	25	0.5505	38	1.8165	125	0.8760	14	10
29 0	0.4848	26	0.5543	38	1.8040	123	0.8746	14	0 61
10	0.4874	25	0.5581	38	1.7917	121	0.8732	14	50
20	0.4899	25	0.5619	39	1.7796	121	0.8718	14	40
30	0.4924	26	0.5658	38	1.7675	119	0.8704	15	30
40	0.4950	25	0.5696	39	1.7556	119	0.8689	14	20
50	0.4975	25	0.5735	39	1.7437	116	0.8675	15	10
30 0	0.5000		0.5774		1.7321		0.8660		0 60
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	149	131	39		38	37	36		25	14	13
.1	14.9	13.1	3.9	.1	3.8	3.7	3.6	.1	2.5	1.4	1.3
.2	29.8	26.2	7.8	.2	7.6	7.4	7.2	.2	5.0	2.8	2.6
.3	44.7	39.3	11.7	.3	11.4	11.1	10.8	.3	7.5	4.2	3.9
.4	59.6	52.4	15.6	.4	15.2	14.8	14.4	.4	10.0	5.6	5.2
.5	74.5	65.5	19.5	.5	19.0	18.5	18.0	.5	12.5	7.0	6.5
.6	89.4	78.6	23.4	.6	22.8	22.2	21.6	.6	15.0	8.4	7.8
.7	104.3	91.7	27.3	.7	26.6	25.9	25.2	.7	17.5	9.8	9.1
.8	119.2	104.8	31.2	.8	30.4	29.6	28.8	.8	20.0	11.2	10.4
.9	134.1	117.9	35.1	.9	34.2	33.3	32.4	.9	22.5	12.6	11.7

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
30 0	0.5000	25	0.5774	38	1.7321	116	0.8660	14	0 **60**
10	0.5025	25	0.5812	39	1.7205	115	0.8646	15	50
20	0.5050	25	0.5851	39	1.7090	113	0.8631	15	40
30	0.5075	25	0.5890	40	1.6977	113	0.8616	15	30
40	0.5100	25	0.5930	39	1.6864	111	0.8601	14	20
50	0.5125	25	0.5969	40	1.6753	110	0.8587	15	10
31 0	0.5150	25	0.6009	39	1.6643	109	0.8572	15	0 **59**
10	0.5175	25	0.6048	40	1.6534	108	0.8557	15	50
20	0.5200	25	0.6088	40	1.6426	107	0.8542	16	40
30	0.5225	25	0.6128	40	1.6319	107	0.8526	15	30
40	0.5250	25	0.6168	40	1.6212	105	0.8511	15	20
50	0.5275	24	0.6208	41	1.6107	104	0.8496	16	10
32 0	0.5299	25	0.6249	40	1.6003	103	0.8480	15	0 **58**
10	0.5324	24	0.6289	41	1.5900	102	0.8465	15	50
20	0.5348	25	0.6330	41	1.5798	101	0.8450	16	40
30	0.5373	25	0.6371	41	1.5697	100	0.8434	16	30
40	0.5398	24	0.6412	41	1.5597	100	0.8418	15	20
50	0.5422	24	0.6453	41	1.5497	98	0.8403	16	10
33 0	0.5446	25	0.6494	42	1.5399	98	0.8387	16	0 **57**
10	0.5471	24	0.6536	41	1.5301	97	0.8371	16	50
20	0.5495	24	0.6577	42	1.5204	96	0.8355	16	40
30	0.5519	25	0.6619	42	1.5108	95	0.8339	16	30
40	0.5544	24	0.6661	42	1.5013	94	0.8323	16	20
50	0.5568	24	0.6703	42	1.4919	93	0.8307	17	10
34 0	0.5592	24	0.6745	42	1.4826	93	0.8290	16	0 **56**
10	0.5616	24	0.6787	43	1.4733	92	0.8274	16	50
20	0.5640	24	0.6830	43	1.4641	91	0.8258	17	40
30	0.5664	24	0.6873	43	1.4550	90	0.8241	16	30
40	0.5688	24	0.6916	43	1.4460	90	0.8225	17	20
50	0.5712	24	0.6959	43	1.4370	89	0.8208	16	10
35 0	0.5736		0.7002		1.4281		0.8192		0 **55**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	43	42	41		40	25	24		17	16	15
.1	4.3	4.2	4.1	.1	4.0	2.5	2.4	.1	1.7	1.6	1.5
.2	8.6	8.4	8.2	.2	8.0	5.0	4.8	.2	3.4	3.2	3.0
.3	12.9	12.6	12.3	.3	12.0	7.5	7.2	.3	5.1	4.8	4.5
.4	17.2	16.8	16.4	.4	16.0	10.0	9.6	.4	6.8	6.4	6.0
.5	21.5	21.0	20.5	.5	20.0	12.5	12.0	.5	8.5	8.0	7.5
.6	25.8	25.2	24.6	.6	24.0	15.0	14.4	.6	10.2	9.6	9.0
.7	30.1	29.4	28.7	.7	28.0	17.5	16.8	.7	11.9	11.2	10.5
.8	34.4	33.6	32.8	.8	32.0	20.0	19.2	.8	13.6	12.8	12.0
.9	38.7	37.8	36.9	.9	36.0	22.5	21.6	.9	15.3	14.4	13.5

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
35 0	0.5736	24	0.7002	44	1.4281	88	0.8192	17	0 **55**
10	0.5760	23	0.7046	43	1.4193	87	0.8175	17	50
20	0.5783	24	0.7089	44	1.4106	87	0.8158	17	40
30	0.5807	24	0.7133	44	1.4019	85	0.8141	17	30
40	0.5831	23	0.7177	44	1.3934	86	0.8124	17	20
50	0.5854	24	0.7221	44	1.3848	84	0.8107	17	10
36 0	0.5878	23	0.7265	45	1.3764	84	0.8090	17	0 **54**
10	0.5901	24	0.7310	45	1.3680	83	0.8073	17	50
20	0.5925	23	0.7355	45	1.3597	83	0.8056	17	40
30	0.5948	24	0.7400	45	1.3514	82	0.8039	18	30
40	0.5972	23	0.7445	45	1.3432	81	0.8021	17	20
50	0.5995	23	0.7490	46	1.3351	81	0.8004	18	10
37 0	0.6018	23	0.7536	45	1.3270	80	0.7986	17	0 **53**
10	0.6041	24	0.7581	46	1.3190	79	0.7969	18	50
20	0.6065	23	0.7627	46	1.3111	79	0.7951	17	40
30	0.6088	23	0.7673	47	1.3032	78	0.7934	18	30
40	0.6111	23	0.7720	46	1.2954	78	0.7916	18	20
50	0.6134	23	0.7766	47	1.2876	77	0.7898	18	10
38 0	0.6157	23	0.7813	47	1.2799	76	0.7880	18	0 **52**
10	0.6180	22	0.7860	47	1.2723	76	0.7862	18	50
20	0.6202	23	0.7907	47	1.2647	75	0.7844	18	40
30	0.6225	23	0.7954	48	1.2572	75	0.7826	18	30
40	0.6248	23	0.8002	48	1.2497	74	0.7808	18	20
50	0.6271	22	0.8050	48	1.2423	74	0.7790	19	10
39 0	0.6293	23	0.8098	48	1.2349	73	0.7771	18	0 **51**
10	0.6316	22	0.8146	49	1.2276	73	0.7753	18	50
20	0.6338	23	0.8195	48	1.2203	72	0.7735	19	40
30	0.6361	22	0.8243	49	1.2131	72	0.7716	18	30
40	0.6383	23	0.8292	50	1.2059	71	0.7698	19	20
50	0.6406	22	0.8342	49	1.1988	70	0.7679	19	10
40 0	0.6428		0.8391		1.1918		0.7660		0 **50**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	48	47	46		45	44	23		22	19	18
.1	4.8	4.7	4.6	.1	4.5	4.4	2.3	.1	2.2	1.9	1.8
.2	9.6	9.4	9.2	.2	9.0	8.8	4.6	.2	4.4	3.8	3.6
.3	14.4	14.1	13.8	.3	13.5	13.2	6.9	.3	6.6	5.7	5.4
.4	19.2	18.8	18.4	.4	18.0	17.6	9.2	.4	8.8	7.6	7.2
.5	24.0	23.5	23.0	.5	22.5	22.0	11.5	.5	11.0	9.5	9.0
.6	28.8	28.2	27.6	.6	27.0	26.4	13.8	.6	13.2	11.4	10.8
.7	33.6	32.9	32.2	.7	31.5	30.8	16.1	.7	15.4	13.3	12.6
.8	38.4	37.6	36.8	.8	36.0	35.2	18.4	.8	17.6	15.2	14.4
.9	43.2	42.3	41.4	.9	40.5	39.6	20.7	.9	19.8	17.1	16.2

° ′	Sin.	d.	Tang.	d.	Cotg.	d.	Cos.	d.	
40 0	0.6428	22	0.8391	50	1.1918	71	0.7660	18	0 **50**
10	0.6450	22	0.8441	50	1.1847	69	0.7642	19	50
20	0.6472	22	0.8491	50	1.1778	70	0.7623	19	40
30	0.6494	23	0.8541	50	1.1708	68	0.7604	19	30
40	0.6517	22	0.8591	51	1.1640	69	0.7585	19	20
50	0.6539	22	0.8642	51	1.1571	67	0.7566	19	10
41 0	0.6561	22	0.8693	51	1.1504	68	0.7547	19	0 **49**
10	0.6583	21	0.8744	52	1.1436	67	0.7528	19	50
20	0.6604	22	0.8796	51	1.1369	66	0.7509	19	40
30	0.6626	22	0.8847	52	1.1303	66	0.7490	20	30
40	0.6648	22	0.8899	53	1.1237	66	0.7470	19	20
50	0.6670	21	0.8952	52	1.1171	65	0.7451	20	10
42 0	0.6691	22	0.9004	53	1.1106	65	0.7431	19	0 **48**
10	0.6713	21	0.9057	53	1.1041	64	0.7412	20	50
20	0.6734	22	0.9110	53	1.0977	64	0.7392	19	40
30	0.6756	21	0.9163	54	1.0913	63	0.7373	20	30
40	0.6777	22	0.9217	54	1.0850	64	0.7353	20	20
50	0.6799	21	0.9271	54	1.0786	62	0.7333	19	10
43 0	0.6820	21	0.9325	55	1.0724	63	0.7314	20	0 **47**
10	0.6841	21	0.9380	55	1.0661	62	0.7294	20	50
20	0.6862	22	0.9435	55	1.0599	61	0.7274	20	40
30	0.6884	21	0.9490	55	1.0538	61	0.7254	20	30
40	0.6905	21	0.9545	56	1.0477	61	0.7234	20	20
50	0.6926	21	0.9601	56	1.0416	61	0.7214	21	10
44 0	0.6947	20	0.9657	56	1.0355	60	0.7193	20	0 **46**
10	0.6967	21	0.9713	57	1.0295	60	0.7173	20	50
20	0.6988	21	0.9770	57	1.0235	59	0.7153	20	40
30	0.7009	21	0.9827	57	1.0176	59	0.7133	21	30
40	0.7030	20	0.9884	58	1.0117	59	0.7112	20	20
50	0.7050	21	0.9942	58	1.0058	58	0.7092	21	10
45 0	0.7071		1.0000		1.0000		0.7071		0 **45**
	Cos.	d.	Cotg.	d.	Tang.	d.	Sin.	d.	′ °

PP	57	55	54		53	51	22		21	20	19
.1	5.7	5.5	5.4	.1	5.3	5.1	2.2	.1	2.1	2.0	1.9
.2	11.4	11.0	10.8	.2	10.6	10.2	4.4	.2	4.2	4.0	3.8
.3	17.1	16.5	16.2	.3	15.9	15.3	6.6	.3	6.3	6.0	5.7
.4	22.8	22.0	21.6	.4	21.2	20.4	8.8	.4	8.4	8.0	7.6
.5	28.5	27.5	27.0	.5	26.5	25.5	11.0	.5	10.5	10.0	9.5
.6	34.2	33.0	32.4	.6	31.8	30.6	13.2	.6	12.6	12.0	11.4
.7	39.9	38.5	37.8	.7	37.1	35.7	15.4	.7	14.7	14.0	13.3
.8	45.6	44.0	43.2	.8	42.4	40.8	17.6	.8	16.8	16.0	15.2
.9	51.3	49.5	48.6	.9	47.7	45.9	19.8	.9	18.9	18.0	17.1

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Author Merrill

Title Elementary M

DATE.	
Oct 24/13	Bell, E. C
~~Nov~~ 1/13.	do
Nov. 14/13	do.
Feb 4/16	Conn
14. V. 28	Harris
1. XI. 30	[illegible]